The General Linear RE Model

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Outline

• The basic idea behind eigenvector decomposition approaches to solving linear RE models

• Canonical forms, continuous and discrete time

• What determines existence and uniqueness

• Allocating effort between yourself and the computer
Our most general canonical form

\[ \Gamma_0 y(t) = \Gamma_1 y(t - 1) + C + \Psi z(t) + \Pi \eta(t), \]

\[ t = 1, \ldots, T. \] (1)

\( C \): a vector of constants
\( z(t) \): an exogenous random disturbance
\( \eta(t) \): an expectational error

All we know about \( \eta(t) \) is that \( E_t \eta(t + 1) = 0 \), all \( t \). The actual values of \( \eta(t) \) have to be determined in solving the model.
Note: No $E_t x(t + 1)$ terms in the system. We’ve replaced any such term by

$$x(t + 1) - (x(t + 1) - E_t x(t + 1)) = x(t + 1) - \eta(t + 1).$$

Convention: Anything dated $t$ is known at $t$, i.e. $E_t x(t) \equiv x(t)$ for any $x$. 

Why a Canonical Form?

• It is some work to get a model into this form. Models often have more than one lag. They often have $z(t)$ and $\eta(t + 1)$ in the same equation. They often have $E_t x(t + s)$ terms with $s > 1$. But for this form, the work is modest.

• Once the model is in a canonical form, the solution set can be described automatically, by the computer.
Example

\[ y_t = -\theta(r_t - E_t \pi_{t+1}) + E_t y_{t+1} + \varepsilon_t \]  \hspace{1cm} (2)

\[ \pi_t = \gamma y_t + \beta E_t \pi_{t+1} + \nu_t \]  \hspace{1cm} (3)

\[ \Gamma_0 = \begin{bmatrix} 1 & \theta \\ 0 & \beta \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix}, \quad \Pi = I_{2 \times 2}, \quad \Phi = \begin{bmatrix} -1 & 0 & \theta \\ 0 & -1 & 0 \end{bmatrix}. \]
What *gensys.m* Produces

- existence: is there any solution?
- uniqueness: is there at most one solution? (Non-existence and non-uniqueness can coexist.)
- completeness: are there as many equations as variables?

\[
y(t) = \Theta_1 y(t - 1) + \Theta_c + \Theta_0 \hat{z}(t) + \Theta_y \sum_{s=1}^{\infty} \Theta_{f}^{s-1} \Theta_z E_t \hat{z}(t + s) \tag{4}
\]

\(\Theta_1: G1\)
$\Theta_c$: C
$\Theta_0$: impact
$\Theta_y$: ywt
$\Theta_f$: fmat
$\Theta_z$: fwt
Impulse responses

Impulse responses trace out the effect on the system of unit increases, lasting only one period, in elements of the $z$ vector. If $z$ is i.i.d., and $y$ is stationary, the impulse responses are also the coefficients of the moving average representation for $y$. If $z$ is i.i.d., the matrix of effects $s$ periods from now on $y$ emerging from unit increases now in $z$ is given by the matrix $\Theta_1^s\Theta_0$, where the rows of the matrix correspond to the elements of $y$ and the columns correspond to the elements of $z$ that are being perturbed. When $z$ is not i.i.d, the impulse responses depend on how expected future $z$’s react to a change in current $z$, and thus can’t be determined without expanding the model to describe explicitly $z$’s serial dependence properties.
Impulse responses are often displayed by plotting the $i, j$'th element of this impulse response matrix as a function of $s$. This is the time path of the response of variable $i$ to a unit disturbance in $z$. Though impulse responses contain no information not available in principle in $\Theta_0$ and $\Theta_1$, they are usually easier to interpret. They display “typical modes of behavior” for variables in the system and fit an “if this happens, then that happens” interpretation.
The Details, for a Simplified Canonical Form

- $\Gamma_0 = I$

- Stability conditions:

$$E_s \left[ \phi_i y(t) \xi_i^{-t} \right] \xrightarrow{t \to \infty} 0, \quad i = 1, \ldots, \infty \tag{5}$$

- Jordan decomposition

$$\Gamma_1 = P\Lambda P^{-1}$$
\( \Lambda \) is “almost diagonal”, with “Jordan blocks” down the diagonal.

\[
\Lambda = \begin{bmatrix}
\Lambda_1 & 0 & \cdots & 0 \\
0 & \Lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Lambda_m
\end{bmatrix}
\]  

(6)

\[
\Lambda_j = \begin{bmatrix}
\lambda_j & 1 & 0 & \cdots & 0 \\
0 & \lambda_j & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_j & 1 \\
0 & 0 & \cdots & 0 & \lambda_j
\end{bmatrix}
\]  

(7)
\( w(t) = P^{-1}y(t) \), so

\[
w(t) = \Lambda w(t - 1) + P^{-1}C + P^{-1}(\Psi z(t) + \Pi \eta(t))
\]

- Consider block \( j \):

\[
w_j(t) = \Lambda_j w_j(t - 1) + P^j \cdot C + P^j \cdot (\Psi z(t) + \Pi \eta(t))
\]
Solving backward yields

\[ w_j(t) = \Lambda_j^t w_j(0) + (I - \Lambda_j)^{-1}(I - \Lambda_j^t)P^j \cdot C \]
\[ + \sum_{s=0}^{t-1} \Lambda_j^s P^j \cdot (\Psi z(t - s) + \Pi \eta(t - s)) \]  

(8)

If \( w_j \) is of length \( m_j \), then the elements of \( \Lambda_j^t \) are products of polynomials in \( t \) of order at most \( m_j \) with \( \lambda_j^t \), where \( \lambda_j \) is the diagonal element of \( \Lambda_j \).

Therefore if there is any \( i \) such that \( \phi_i P^j \neq 0 \) and \( \lambda_j \geq \xi_i \), the only solution for \( w_j \) that satisfies the stability conditions is the forward
solution

\[ w_j(t) = (I - \Lambda_j)^{-1} P^j \cdot C - \sum_{s=1}^{\infty} \Lambda_j^{-s} P^j \cdot E_t[\Psi z(t + s)] \]

- In the special case where \( E_t z(t + 1) \equiv 0 \), the last term drops and \( w_j \) must be a constant. But from (8), \( w_j(t) \) has in this case one-step-ahead prediction error (innovation)

\[ P^j \cdot (\Psi z(t) + \Pi \eta(t)) = 0 . \]  

(9)

- For every \( j \) whose root needs to be “suppressed”, we get such an equation. Stacking up the corresponding \( P^j \)'s into a matrix \( P^u \) (u
for “unstable”), we get

\[ P^u \Psi z(t) = -P^u \Pi \eta(t). \quad (10) \]

- If the space spanned by the columns of \( P^u \Pi \) includes all the columns of \( P^u \Psi \), then for every possible \( z(t) \) we can solve for \( \eta(t) \) from (10). This is the condition for existence of a solution. Notice that it depends on the idea that the \( z(t) \) vectors are unrestricted.

- If the space spanned by the rows of \( P^u \Pi \) contains all the rows of \( P^s \Pi \), where \( P^s \) is the matrix formed from all the rows of \( P^{-1} \) not contained in \( P^u \), then the value of \( P^u \Pi \eta(t) \) determined by (10) also determines the value of \( P^s \Pi \eta(t) \), and we have uniqueness.