The General Linear RE Model

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Outline

- The basic idea behind eigenvector decomposition approaches to solving linear RE models
- Canonical forms, continuous and discrete time
- What determines existence and uniqueness
- Allocating effort between yourself and the computer

Our most general canonical form

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi\eta(t),$$

$$t = 1, \ldots, T \,. \quad (1)$$

C: a vector of constants
$z(t)$: an exogenous random disturbance
$\eta(t)$: an expectational error

All we know about $\eta(t)$ is that $E_t\eta(t+1) = 0$, all t. The actual values of $\eta(t)$ have to be determined in solving the model.

Note: No $E_t x(t+1)$ terms in the system. We've replaced any such term by

$$x(t+1) - \big(x(t+1) - E_t x(t+1)\big) = x(t+1) - \eta(t+1)\ .$$

Convention: Anything dated t is known at t, i.e. $E_t x(t) \equiv x(t)$ for any x.

Why a Canonical Form?

- It is some work to get a model into this form. Models often have more than one lag. They often have $z(t)$ and $\eta(t+1)$ in the same equation. They often have $E_t x(t+s)$ terms with $s > 1$. But for this form, the work is modest.

- Once the model is in a canonical form, the solution set can be described automatically, by the computer.

Example

$$y_t = -\theta(r_t - E_t\pi_{t+1}) + E_t y_{t+1} + \varepsilon_t \tag{2}$$

$$\pi_t = \gamma y_t + \beta E_t \pi_{t+1} + \nu_t \tag{3}$$

$$\Gamma_0 = \begin{bmatrix} 1 & \theta \\ 0 & \beta \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix}, \quad \Pi = \underset{2\times 2}{I},$$

$$\Phi = \begin{bmatrix} -1 & 0 & \theta \\ 0 & -1 & 0 \end{bmatrix}.$$

What `gensys.m` Produces

- existence: is there any solution?
- uniqueness: is there at most one solution? (Non-existence and non-uniqueness can coexist.)
- completeness: are there as many equations as variables?

$$y(t) = \Theta_1 y(t-1) + \Theta_c + \Theta_0 z(t) + \Theta_y \sum_{s=1}^{\infty} \Theta_f^{s-1} \Theta_z E_t z(t+s) \quad (4)$$

Θ_1: `G1`

Θ_c: `C`

Θ_0: `impact`

Θ_y: `ywt`

Θ_f: `fmat`

Θ_z: `fwt`

Impulse responses

Impulse responses trace out the effect on the system of unit increases, lasting only one period, in elements of the z vector. If z is i.i.d., and y is stationary, the impulse responses are also the coefficients of the moving average representation for y. If z is i.i.d., the matrix of effects s periods from now on y emerging from unit increases now in z is given by the matrix $\Theta_1^s\Theta_0$, where the rows of the matrix correspond to the elements of y and the columns correspond to the elements of z that are being perturbed. When z is not i.i.d, the impulse responses depend on how expected future z's react to a change in current z, and thus can't be determined without expanding the model to describe explicitly z's serial dependence properties.

Impulse responses are often displayed by plotting the i, j'th element of this impulse response matrix as a function of s. This is the time path of the response of variable i to a unit disturbance in z. Though impulse responses contain no information not available in principle in Θ_0 and Θ_1, they are usually easier to interpret. They display "typical modes of behavior" for variables in the system and fit an "if this happens, then that happens" interpretation.

The Details, for a Simplified Canonical Form

- $\Gamma_0 = I$

- Stability conditions:

$$E_s\left[\phi_i y(t) \xi_i^{-t}\right] \underset{t \to \infty}{\longrightarrow} 0, \quad i = 1, \ldots \infty \tag{5}$$

- Jordan decomposition

$$\Gamma_1 = P \Lambda P^{-1}$$

Λ is “almost diagonal”, with “Jordan blocks” down the diagonal.

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_m \end{bmatrix} \tag{6}$$

$$\Lambda_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix} \tag{7}$$

- $w(t) = P^{-1}y(t)$, so

$$w(t) = \Lambda w(t-1) + P^{-1}C + P^{-1}\big(\Psi z(t) + \Pi\eta(t)\big)$$

- Consider block j:

$$w_j(t) =$$

$$\Lambda_j w_j(t-1) + P^{j\cdot}C + P^{j\cdot}\big(\Psi z(t) + \Pi\eta(t)\big)$$

- Solving backward yields

$$w_j(t) = \Lambda_j^t w_j(0) + (I - \Lambda_j)^{-1}(I - \Lambda_j^t)P^{j\cdot}C + \sum_{s=0}^{t-1} \Lambda_j^s P^{j\cdot}(\Psi z(t-s) + \Pi\eta(t-s)) \quad (8)$$

- If w_j is of length m_j, then the elements of Λ_j^t are products of polynomials in t of order at most m_j with λ_j^t, where λ_j is the diagonal element of Λ_j.

- Therefore if there is any i such that $\phi_i P^{j\cdot} \neq 0$ and $\lambda_j \geq \xi_i$, the only solution for w_j that satisfies the stability conditions is the forward solution

$$w_j(t) = (I - \Lambda_j)^{-1} P^{j\cdot} C - \sum_{s=1}^{\infty} \Lambda_j^{-s} P^{j\cdot} E_t[\Psi z(t+s)]$$

- In the special case where $E_t z(t+1) \equiv 0$, the last term drops and w_j must be a constant. But from (8), $w_j(t)$ has in this case one-step-ahead prediction error (innovation)

$$P^{j\cdot}(\Psi z(t) + \Pi \eta(t)) = 0\,. \tag{9}$$

- For every j whose root needs to be "suppressed", we get such an equation. Stacking up the corresponding $P^{j\cdot}$'s into a matrix P^u (u

for "unstable"), we get

$$P^u \Psi z(t) = -P^u \Pi \eta(t) \,. \tag{10}$$

- If the space spanned by the columns of $P^u\Pi$ includes all the columns of $P^u\Psi$, then for every possible $z(t)$ we can solve for $\eta(t)$ from (10). This is the condition for existence of a solution. Notice that it depends on the idea that the $z(t)$ vectors are unrestricted.

- If the space spanned by the *rows* of $P^u\Pi$ contains all the rows of $P^s\Pi$, where P^s is the matrix formed from all the rows of P^{-1} not contained in P^u, then the value of $P^u\Pi\eta(t)$ determined by (10) also determines the value of $P^s\Pi\eta(t)$, and we have uniqueness.