FINAL EXAM

This a three hour (180 minute) exam. There are 180 points on the exam, so you should allocate about as many minutes to each question as it has points. Points are specified for each of the questions. You are to answer all questions, and partial credit is given for partial answers.

(1) (60 points) Consider a discrete time search model in which workers who have a job inperiod t paying the wage w_t consume their wage and receive utility $u(w_t)$, where u is a function with $u'(w) > 0$, $u''(w) \le 0$ for all $w > 0$. Workers discount future utility at the rate $\beta < 1$. While working at a given job, they receive the same wage each period. Each period there is a probability δ that their job will end and they will become unemployed. While unemployed, they receive an unemployment benefit $b \ge 0$ each period, and each period while unemployed there is a prbability θ that they will receive a job offer. Job offers are drawn from a fixed distribution over a finite number of possible wages. The wage values are $w_1, \dots, w_n$ and they have probabilities $\pi_1, \dots, \pi_n$.

(a) Write recursive equations for discounted expected utility in the states of being unemployed and of being employed at wage w_i, $i = 1, \dots, n$.
Let U be the continuation value if one is unemployed in the current period and $W(w_i)$ be the continuation value if one is employed at the wage w_i. Then

$$W(w_i) = u(w_i) + \beta\big((1-\delta)W(w_i) + \delta U\big)$$
$$U = u(b) + \beta\Big((1-\theta)U + \theta\sum_i \pi_i \max(W(w_i), U)\Big)$$

from which we can derive

$$W(w_i) = \frac{u(w_i) + \beta\delta U}{1 - \beta(1-\delta)}$$
$$U = \frac{u(b) + \theta\sum_i \pi_i max(W(w_i) - U, 0)}{1-\beta}\,.$$

(b) Prove that a reservation wage strategy, i.e. a strategy that accepts any job at a wage above a certain cutoff and refuses jobs below the cutoff, is optimal in this case.
It is obvious that if one has a job offer at wage w, it is optimal to take it if and only if $W(w) > U$. The only issue, then, is whether the ranking of jobs by $W(w)$ matches the ranking of jobs by w, so that the "reservation $W(w)$"

corresponds to a reservation wage. From the formula for $W(w_i)$ above, we see that with U fixed, and since $u' > 0$ by assumption, $W(w_i)$ is increasing in w_i. Of course if we change the w_i distribution we change U, but here we are not considering varying the distribution; we are considering how, with the distribution of wages fixed, $W(w_i)$ varies with i.

(c) Describe how the equations you have written down could be used to solve for the reservation wage.

With the reservation wage fixed between i and $i+1$, the two equations are simply linear equations in U and $W(w_i)$, $i = 1, \ldots, n$. So we can try each interval, within each interval solving for this vector of continuation values. The reservation wage $\bar{w}$ is then defined by $W(\bar{W}) = U$, so we can find it by solving

$$U = \frac{u(\bar{w}) + \beta\delta U}{1 - \beta(1-\delta)}$$

for $\bar{w}$. When we have found the right interval $i, i+1$, $\bar{w}$ will lie between w_i and w_{i+1}.

(d) If $u'' \equiv 0$, an economy in which each agent solved this search optimization problem would be allocating labor optimally, if we take the search technology (i.e. $\vec{\pi} = \{\pi_i\}$, $\vec{w} = \{w_i\}$, δ and θ) as a given constraint. But if $u'' < 0$, the resulting equilibrium is generally not socially optimal. Explain why.

This question left it up to you to clarify what social optimality might mean here (which bothered some people). If it means maximizing expected discounted utility summed across agents subject to the search technology, then as in the problem set, with linear utility ($u'' = 0$), agents will be optimizing and no planner's reallocations can improve matters. This depends on b representing a true "utility of leisure". If b is provided by taxing those working, $b > 0$ will imply inefficiency, because the taxes and transfers will make working less attractive relative to not working.

With risk aversion ($u'' < 0$), the agents will dislike staying unemployed relative to taking a job, because the former implies a riskier stream of income. If we assume that there are many workers searching indpendently, a social planner would like them to maximize the discounted expected sum of future wages. In steady state total production (i.e. total wage income) is constant and can be divided equally among teh population. A really excellent answer (of which there were none) would have noted that the problem statement did not specify that workers were searching independently. If there is perfect dependence, so all workers are unemployed or working at the same time (unemployment rate is 100% or zero), the competitive equilibrium is the optimum even with risk aversion. This is a case of aggregate, uninsurable risk.

(e) Would your answer to 1d change if the agents could save, i.e. if they could put aside some of their wage earnings to be consumed later, in case they find themselves unemployed or in a lower-paying job?
In the search i.i.d. across agents case, this would allow consumption smoothing, but not perfect risk sharing, so a planner could still improve on the competitive equilibrium.

(2) (60 points) Consider an economy in which a representative agent solves

$$\max_{C,M,B} E\left[\sum_{t=0-}^{\infty} \beta^t \log(C_t)\right] \qquad \text{subject to} \tag{1}$$

$$C_t(2 - e^{-v_t}) + \frac{B_t + M_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} + Y_t \tag{2}$$

$$v_t = \frac{P_t C_t}{M_t}\,. \tag{3}$$

The government has the budget constraint

$$\frac{B_t + M_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t}\,, \tag{4}$$

and sets the interest rate R_t and primary surplus τ_t according to the following (monetary and fiscal) rules:

$$M_t \equiv \bar{M} \tag{5}$$

$$\tau_t = -\phi_0 + \phi_1 b_{t-1}\,, \tag{6}$$

where ϕ_0 and ϕ_1 are both positive.

(a) Show that the agent's constraint implies that transactions costs — the gap between spending on consumption goods and utility-yielding consumption — are increasing in velocity v, but at a decreasing rate.
If transactions costs are to be a function of v alone, it must be the relative gap between spending on consumption and C, that is transactions costs, i.e. $(2 - e^{-v}) - 1$. The derivative of this with respect to v is $e^{-v} > 0$, and the second derivative is $-e^{-v} < 0$.

(b) Show that under certain conditions on the parameters the economy has a competitive equilibrium in which prices are stable or stably varying.
The GBC given in this problem used b_{t-1}, which was not defined. (It was intended, following the literature, to be B_{t-1}/P_{t-1}.) However, until the last part of the problem, where active fiscal, passive money, was introduced and replaced this fiscal rule, the analysis of existence and uniqueness did

not depend on the fiscal rule, so long as it was passive. The FOC's are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t(2 - e^{-v_t} + v_t e^{-v_t})$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = \beta R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right]$$

$$\partial M: \qquad \frac{\lambda_t}{P_t}(1 - v_t^2 e^{-v_t}) = \beta E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right].$$

From the B and M FOC's we can derive $R_t = (1 - v_t^2 e^{-v_t})^{-1}$. Letting $z_t = \lambda_t / P_t$, we see that

$$z_t = \frac{1}{\bar{M} v_t(2 - e^{-v_t} + v_t e^{-v_t})}$$

is a function of v_t alone under the fixed-M monetary policy, and the M FOC can be written as

$$z_t(1 - v_t^2 e^{-v_t}) = \beta E_t z_{t+1}\,. \qquad (*)$$

This difference equation has as one solution, then, any value of z_t that corresponds to a $v_t = \bar{v}$ with

$$1 - \bar{v}^2 e^{-\bar{v}} = \beta\,. \qquad (\dagger)$$

since $v^2 e^{-v}$ is bounded, with a maximum at $v = 2$, where it is $4e^{-2} = .54$ (you were not expected to get the number on the exam, just the point that there is a bound), there will be such a steady state $\bar{v}$ for any $\beta \in (.46, 1)$. Notice, though, that except right at $\beta = .54$, there are generally two solutions for $\bar{v}$ for any given β. One student correctly noted the bound on β. No student noticed that there are two $\bar{v}$'s for any given β.

With v constant, we then know that $P_t C_t$ is constant because of the constant-M policy. We also know that C_t fluctuates in proportion to Y_t, form the social resource constraint $Y_t = C_t(2 - e^{-v_t})$ (derived from the private budget constraint and the government budget constraint. Then if $R_t \equiv (1 - \bar{v}^2 e^{-\bar{v}})^{-1}$, all the FOC's and constraints are satisfied. A perfect score could have been obtained by stopping here, noting that there is at least one steady-state solution to FOC's and constraints of this form, and generally two.

However, this setup, despite its having apparently reasonable behavior of transactions costs as determined in the first part of the problem, makes the budget constraint non-convex in M at some values of v. In particular, for large enough v, the second derivative of the constraint with respect to M turns positive. Thus, some solutions to the FOC's are not optima, and there is no guarantee that local optima are in fact global optima. There is here, as in many such models with a fixed-M policy, an equilibrium with $P \equiv v \equiv \infty$, so money is valueless. Not only does this equilibrium exist, but it results in finite utility and positive consumption, and because of the non-convexity of the constraint set, we cannot be sure without further analysis that this equilibrium is worse than a local optimum that satisfies the FOC's.

Since no one even noticed that there are two $\bar{v}$'s for every β, also no one noticed that this is an example where FOC's alone are not enough to pin down a solution.

(c) Under what conditions if any is the initial price level uniquely determined?

This is a hard question, and you were expected only to recognize that making the usual arguments work was difficult here. It can be verified that z is monotone decreasing in v, converging to 0 and ∞ as v goes to ∞ and 0. But the $1 - v_t e^{-v_t}$ factor, which corresponds to terms in our classroom models that were monotone in v, is no longer monotone in v. Solutions to the z_t equation $(*)$ are indeed by the usual arguments *locally* unstable around the *lower* of the two solutions for $\bar{v}$, but if v starts out above that level, it tends to rise toward the upper solution, where it sticks. That is, the usual argument that if v_t deviates above $\bar{v}$ it must have non-zero probability of rising arbitrarily high no longer applies. Thus it is unlikely that the price level is uniquely determined, though a precise answer to this would depend on accounting for the implications of the non-convexity.

(d) Repeat your analysis if the policy equations (5)-(6) were instead $R_t = \bar{R}$ and $\tau_t = \bar{\tau}$.

In our classroom and notes models, pegging $R_t = \bar{R}$ produces a fixed $v_t = \bar{v}$ directly from FOC's. Here it only fixes v_t at one of the two solutions to (†). If v_t remains fixed at either value of $\bar{v}$, the GBC plus the policy rule of fixed $\bar{\tau}$ imply a uniquely determined price level by the usual arguments: the GBC, solved forward in expectation, implies a unique value for the real value of the debt. When this is plugged in to the GBC at time zero, it delivers a unique intial price level. Howver, the two steady-state $v's$ imply two different price levels, because they imply different steady state inflation rates, and hence seignorage levels. But the second steady-state v is in the region where the constraint is not convex, so it does not represent an optimum for the agent, and the agent will not choose it. This does not eliminate the possibility that the nonconvexity could imply that the $M_t = 0$ solution might be better for the private agent than the valued-money equilibrium, but it does imply that the solution will not involve an infinity of possible initial prices.

(3) (60 points) Consider the following model, which has been log-linearized around steady state:

$$\bar{r} = \sigma E_t[y_{t+1} - y_t] + E_t\pi_{t+1} \tag{7}$$

$$b_t = \bar{r}b_{t-1} - \pi_t\bar{b} - \bar{\tau} \tag{8}$$

$$y_t = \rho y_{t-1} + \varepsilon_t \tag{9}$$

b is real government debt, y is output (and consumption, since there is no capital), r is the gross nominal interest rate, π is inflation, and τ is the primary surplus.

(a) Show that, if we rule out unstable solutions and parameters are in certain reasonable ranges, the model has a uniquely determined initial inflation rate.

The constant terms in these equations were not specified carefully. Since they include constants, presumably the variables are not all deviations from steady state, yet the first equation implies a deterministic steady state real gross interest rate of 1.0, and the budget constraint implies that $\bar{b}$ is not actually the steady state value of debt. The equations remain internally consistent, though, so the questions had straightforeward answers.

Using the first equation in the second and taking E_{t-1} of the equation, to eliminate π_t, we get

$$E_{t-1}b_t = \bar{r}b_{t-1} - \bar{b}\bar{r} + \bar{b}(E_{t-1}y_t - y_{t-1})\,.$$

The third equation tells us that y is exogenous and that $E_t y_{t+1} = \rho y_t$. Therefore we can solve the system forward to obtain, assuming $\bar{r}^{-t}b_t \to 0$, to obtain

$$b_t = \frac{\bar{r}\bar{b}}{\bar{r}-1}\left(1 + (1-\rho)y_t\right) + \frac{\bar{\tau}}{\bar{r}-1}\,.$$

Thus b_t is determined by constants and the exogenously determined y_t. Plugging this value for b_0 into the original budget constraint, we get an equation that determines π_0 from b_{-1}, constants, and y_t.

(b) Suppose now we replace equation (9) above by

$$\pi_t = \theta E_t\pi_{t+1} + \gamma y_t\,. \tag{10}$$

Display the matrices you would obtain in putting this version of the system into a form that could be analyzed for uniqueness and existence of the solution by `gensys` or a similar program.

This question was close to a free gift of points, as it is just translating equations into matrix notation. Letting $x_t = [\pi_t, y_t, b_t]'$, we can write the system as

$$\begin{bmatrix} 1 & \sigma & 0 \\ \bar{b} & 0 & 1 \\ \theta & 0 & 0 \end{bmatrix} x_t = \begin{bmatrix} 0 & \sigma & 0 \\ 0 & 0 & \bar{r} \\ 1 & -\gamma & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \eta_t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \varepsilon_{\tau t} \,. \tag{11}$$

I've added the $\varepsilon_{\tau t}$ in the second equation to reflect the fact that we are considering "disturbing" $\bar{\tau}$. If we were only considering a permanent, fully anticipated change in τ, nothing in the system would change except b.

(c) One of these systems implies that a change in $\bar{\tau}$ will change output, while the other does not. Which system is which, and why do they have these different implications?

What I was looking for here was recognition that the difference between the two systems is that one is a stripped down flex-price model, in which output evolves exogenously, with no dependence on what happens to debt or inflation, while the other replaces the exogenous output assumption with a simple New Keynesian Phillips curve, which is the standard mechanism in many models for introducing non-neutrality, so that nominal disturbances and fiscal policy affect real output. It was easy to observe that the first system makes output exogenous and so insures that nothing else in the model has an impact on output. Showing analytically that a shock to τ must affect output is beyond what you could do on an exam, since it would require solving to suppress the two unstable roots, which is too much calculation even in a 3×3 system. Many attempted to show that τ must affect output in the second system without solving it, but I think there is no correct way to make such an argument without finding a solution.