LINEARIZING AND SOLVING A SIMPLE FTPL MODEL

1. The model: agent's problem

A representative agent solves

$$\max_{C,B} E\left[\sum_{t=0}^{\infty} \beta^t \log(C_t)\right] \qquad \text{subject to}$$

$$C_t + \frac{B_t}{P_t} = R_{t-1}\frac{B_{t-1}}{P_t} - \tau_t + Y_t \,,$$

where the agent is choosing consumption and bond holdings (C and B) while taking P, R, and Y time series as unaffected by the agents own choices. (In equilibrium they are affected by the aggregate values of the choice variables, but each individual assumes that he could deviate from the aggregate behavior without affecting equilibrium prices.)

2. Government behavior

$$\text{GBC}: \qquad \frac{B_t}{P_t} + \tau_t = R_{t-1}\frac{B_{t-1}}{P_t}$$

$$\text{Monetary policy}: \qquad R_t = \beta^{-1}\left(\frac{P_t}{P_{t-1}}\right)^{\theta} \varepsilon_{Mt}$$

$$\text{Fiscal policy}: \qquad \tau_t = -\phi_0 + \phi_1\frac{B_{t-1}}{P_{t-1}} + \varepsilon_{Ft} \,.$$

3. Agent FOC's

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = \beta R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] \,.$$

Multiply B FOC by P_t/λ_t, eliminate λ, to arrive at

$$1 = \beta R_t E_t\left[\frac{P_t C_t}{P_{t+1}C_{t+1}}\right] \,.$$

4. Change variables, simplify

Let $b_t = B_t/P_t$ (real debt). Let $\pi_t = P_t/P_{t-1}$ (gross inflation rate). Recognize that subtracting GBC from agent constraint gives us $C_t = Y_t$ as a social resource constraint. (Warning: the SRC is not a constraint on individual behavior. It is an equilibrium condition. So it is important to use the original private constraint in deriving the private agents' FOC's.) We arrive at

$$C_t = Y_t \tag{1}$$

$$b_t + \tau_t = R_{t-1}\frac{b_{t-1}}{\pi_t} \tag{2}$$

$$R_t = \beta^{-1}\pi_t^{\theta}\varepsilon_{Mt} \tag{3}$$

$$\tau_t = -\phi_0 + \phi_1 b_{t-1} \tag{4}$$

$$1 = \beta R_t E_t\left[\frac{C_t}{\pi_{t+1}C_{t+1}}\right] . \tag{5}$$

This gives us five equations in five unknowns (though of course with leads and lags).

5. Determining the price level

Much of macro theory proceeds with a system like this as follows. Note that, if we replace C by Y using the SRC, (3) and (5) become two equations in only two unknowns, R and π. Perhaps they by themselves can be solved as a RE system to produce a unique equilibrium path for R and π. (Note that this actually would determine the price *level* at time 0, because as of time 0 P_{-1} is given data. Thus determining π_0 uniquely also determines P_0 uniquely.) We proceed to linearize this system around its steady state to ask whether the linear system has a uniquely determined non-explosive solution.

6. Linearizing

It's usually convenient to linearize with respect to logs of variables that are inherently positive. That is, for these variables the first-order Taylor expansion uses derivatives with respect to logged variables and the resulting system is linear in the logs of these variables. In this model, C, Y, π, and R are inherently positive. Our formula for fiscal policy suggests τ could go negative, however, and in that case in principle b could do so also. So we simply linearize w.r.t. those variables.

To begin linearizing, we have to pick a point about which to form the Taylor expansion. The Taylor expansion can only remain accurate if we stay near the point about which we are linearizing. Since the model is stochastic, we will not usually be converging to any one point, but if the shocks are small, the model is stable, and we choose a **deterministic steady state** as the point around which to linearize, the linearization has a chance of being accurate. To solve for a deterministic steady state, we set all stochastic disturbances at their mean values (or medians — it doesn't matter, since medians and means must be close if the shocks are small with high probability), erase the time subscripts from the variables, and solve the resulting system.

7. Equation system for steady state

We assume the mean of Y is $\bar{Y} > 0$ and that ε_M and ε_F have means 1 and 0, respectively.

$$R = \pi^{\theta}\beta^{-1}$$
$$1 = \beta R \,.$$

Thus in steady state $R = \beta^{-1}$ and $\pi = 1$.

8. The linearized system

Linearizing around steady state we get

$$\tilde{R}_t = \theta\tilde{\pi}_t + \tilde{\varepsilon}_{Mt} \tag{6}$$
$$0 = \tilde{R}_t + E_t[\tilde{C}_t - \tilde{C}_{t+1} - \tilde{\pi}_{t+1}] \,. \tag{7}$$

The $\sim$'s indicate log deviations from steady state.

9. Unstable equation

Here it is easy to separate stable and unstable roots by hand. If we eliminate R_t from the system, and use the SRC to replace C_t with Y_t, we get

$$\theta\tilde{\pi}_t + \tilde{\varepsilon}_{Mt} = E_t[\tilde{Y}_{t+1} - \tilde{Y}_t - \tilde{\pi}_{t+1}] \,.$$

This is an unstable difference equation in π_t if $\theta > 1$. Under that assumption, the "backward solution" does not have a meaningful limiting form as we iterate it backward to infinity, but the "forward solution" does. The forward solution here is

$$\tilde{\pi}_t = E_t\left[\sum_{s=0}^{\infty} -\theta^{-s-1}\varepsilon_{M,t+s} + \theta^{-s-1}(\tilde{Y}_{t+s+1} - \tilde{Y}_{t+s})\right] .$$

Now suppose that Y_t and ε_{Mt} are i.i.d. with $E\tilde{Y}_t = 0 = E\tilde{\varepsilon}_{Mt}$. Then most of the expectational terms on the right of the expression above drop out, leaving us with

$$\tilde{\pi}_t = -\theta^{-1}(\tilde{\varepsilon}_{Mt} + \tilde{Y}_t) \,,$$

which tells us that surprise increases in monetary tightness or surprise increases in endowment income reduce the inflation rate.

To complete the solution, we use our solution for π_t to solve for R_t:

$$R_t = -\tilde{Y}_t \,.$$

What about the expectational error in the second equation of the system? It is $\tilde{C}_{t+1} + \tilde{\pi}_{t+1} - E_t[\tilde{C}_{t+1} + \tilde{\pi}_{t+1}]$, which we can see now is just $(1-\theta^{-1})(Y_{t+1} - E_tY_{t+1}) - \theta^{-1}\varepsilon_{Mt}$.

10. Interpreting the solution

- The impulse responses are simple. $\tilde{R}$ and $\tilde{\pi}$ are i.i.d. with mean zero, so there are no delayed effects of shocks.
- R is unaffected by ε_M shocks. That is, even though we write the monetary policy equation with R on the left, as if we were thinking ε_M is a random monetary tightening, these random shifts in monetary policy have no effect on the interest rate. They just directly impact the inflation rate.
- So a least squares regression of R on π has no chance of recovering θ. (There is a large literature of econometric estimates of monetary policy behavioral equations that use least squares with R on the left-hand side.)
- An econometrician who knew the structure of the economy could nonetheless easily estimate θ, under the reasonable assumption that ε_M and $\tilde{Y}$ are mutually independent: estimate a least squares regression of π on R; the coefficient on R would be a consistent estimate of θ^{-1}.
- This is not meant to be a set of universally applicable lessons about estimating monetary policy rules. The general lesson is that interpreting data behaviorally or causally always requires identifying assumptions, and that putting an individual agent's choice variable on the left, and things the agent takes as given on the right, does not produce a linear regression equation when the data come from an equilibrium model.

11. Connection to `gensys`

This system is in `gensys` canonical form, with

$$\Gamma_0 = \begin{bmatrix} 1 & -\theta \\ 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We arrived at this system by substituting Y for C in the system (6-7) and lagging the second of these equations once. We are treating the y_t vector as $[\tilde{R}_t, \tilde{\pi}_t]'$ and the exogenous disturbance vector z_t as $[\varepsilon_{Mt}, \tilde{Y}_{t-1} - E_{t-1}\tilde{Y}_t]'$. Note that in this form the system does not satisfy the condition $E_{t-1}z_t = 0$, so to get correct results out of `gensys` we need to use the forward part of the solution, recognizing that $E_t z_{t+1} = [0, \tilde{Y}_t]'$. It might be good practice for you to give `gensys` (or other RE system solver of your choice) these inputs and make sure that you get existence and uniqueness and the same solution we derived above.

If you wanted to avoid the need to use the forward part of the solution, you could add Y_t to the variable vector and include a third equation $\tilde{Y}_t = \varepsilon_{Yt}$. Then there would be no lagged exogenous variable in the second equation, only a lagged "endogenous" variable. The solution would remain the same, but it would no longer require using the forward part of the `gensys` output.

12. Exercise due Thursday, 2/19

Replace the monetary policy and fiscal policy equations (3-4) by

$$\text{Monetary Policy :} \qquad R_t = \bar{R}\varepsilon_{Mt} \tag{8}$$

$$\text{Fiscal Policy :} \qquad \tau_t = \bar{\tau} + \varepsilon_{Ft}\,. \tag{9}$$

Linearize the whole five-equation system around its steady state and use `gensys` or some other RE solver to compute the solution, checking whether existence and uniqueness hold. Use $\beta = .9$, $\bar{R} = \beta^{-1}$, $\bar{\tau} = 1$. Compare the impulse responses of π_t to ε_M, ε_F and Y_t shocks to what was found above for the two-equation system. What would happen in this system with policy given by (8-9) if we tried the previous strategy of using the monetary policy equation and the FOC to form a two-equation system and tried to solve it in isolation?

13. Looking at the full nonlinear model: policy equations

We can derive neat analytic solutions of the full nonlinear version of this model. The specifications of policy given above, though, produce less than perfectly neat analytic solutions. So here we consider somewhat different pairs of monetary and fiscal policies:

$$\text{Taylor Rule :} \qquad R_t = \beta^{-1}\left(\frac{P_t Y_t}{P_{t-1}Y_{t-1}}\right)^{\theta}\varepsilon_{Mt}$$

$$\text{Fiscal Rule :} \qquad \tau_t = -\phi_0 + \phi_1\frac{B_t}{P_t} + \varepsilon_{Ft}$$

The differences from the policy rules we considered above are that the interest rate in the Taylor rule depends on both inflation and output growth, with the same coefficient θ on each, instead of on inflation alone, and the primary surplus in the fiscal rule depends on current, rather than lagged, real debt. Estimated monetary reaction functions do not usually constrain response of the interest rate to output growth and inflation to be identical, but both tend to be positive, and a policy that like this one that responds to nominal income growth is sometimes proposed as approximately optimal. The change in timing of response to real debt makes no real difference to the model's interpretation; actual fiscal behavior is much more complicated than this.

We will consider an active money form of the Taylor rule, with $\theta > 1$ (a "Taylor Principle" Taylor rule), a passive money form of it, with $\theta = 0$ (i.e., $R_t \equiv \bar{R}$), a passive fiscal rule with $\phi_0 < 0$, $\phi_1 > \beta^{-1} - 1$, and an active fiscal rule with $\phi_0 = -\bar{\tau}$, $\phi_1 = 0$. These assumptions all match, in the linearized model, Leeper's definitions of active and passive policies.

14. The equation system

The private sector constraint and FOC's, and the SRC, are as before, so the equation system is

$$C_t = Y_t \tag{10}$$

$$b_t + \tau_t = R_{t-1}\frac{b_{t-1}}{\pi_t} \tag{11}$$

$$R_t = \beta^{-1}\pi_t^{\theta}\left(\frac{Y_t}{Y_{t-1}}\right)^{\theta}\varepsilon_{Mt} \tag{12}$$

$$\tau_t = -\phi_0 + \phi_1 b_t \tag{13}$$

$$1 = \beta R_t E_t\left[\frac{C_t}{\pi_{t+1}C_{t+1}}\right] . \tag{14}$$

15. Some algebra

Now substituting the expression for R_t from (12) into (14), we arrive at

$$1 = \pi_t^{\theta}\left(\frac{Y_t}{Y_{t-1}}\right)^{\theta}\varepsilon_{Mt}E_t\left[\frac{C_t}{\pi_{t+1}C_{t+1}}\right] . \tag{15}$$

Letting $z_t = \left(Y_{t-1}/(\pi_t Y_t)\right)^{\theta}$ and using the SRC to eliminate C, we can rewrite this as

$$z_t\varepsilon_{Mt}^{-1} = E_t[z_{t+1}^{\theta^{-1}}] . \tag{16}$$

16. Solving under the Taylor Principle

Equation (16) has one solution in which z_t/ε_{Mt} is a constant, call it κ. The right-hand side of (16) is on this path $\kappa^{\theta^{-1}}E_t[\varepsilon_{Mt}^{\theta^{-1}}]$. Since that has to be equal to κ, we can (using the i.i.d. assumptionn on ε_{Mt}) to solve for $\kappa = \left(E[\varepsilon_{Mt}^{\theta^{-1}}]\right)^{\theta/(\theta-1)}$. Now define $w_t = z_t/(\kappa\varepsilon_{Mt})$. By construction w_t has deterministic steady state value 1. We can rewrite the equation once more, now in terms of w:

$$w_t = E_t[W_{t+1}^{1/\theta}\varepsilon_{M,t+1}^{1/\theta}\kappa^{\theta^{-1}-1}] . \tag{17}$$

It can easily be checked that $E_t[\varepsilon_{M,t+1}\kappa^{\theta^{-1}-1}] = 1$. We'll now show that if w_t ever deviates above its steady-state value of 1, it must then have a non-zero probability of growing arbitrarily large. A similar argument would show that if it ever deviates below its steady state value, it must have a non-zero probability of growing arbitrarily close to zero.

Suppose $w_t > 1$. Then there must be a non-zero probability of $w_{t+1} \geq w_t^{\theta}$. For if this were not true, then the full expectation on the right-hand-side would be the product of a random variable with expectation one and a random variable that is with probability one less than w_t. The expectation on the right-hand-side would then have to be less than w_t, which contradicts the assertion in the equation. But we can repeat this argument at $t+1$, etc., reaching the conclusion that with non-zero probabiity $w_{t+s} > w_t^{\theta^s}$. Since $w_t > 1$ and $\theta > 1$, this means w_{t+s} must with positive probability grow to exceed any fixed upper bound.

So we can conclude that when $\theta > 1$, there is only one solution consistent with z_t (and hence π_t) remaining bounded away from zero and ∞, the one where $z_t = \kappa\varepsilon_{Mt}$. This solution implies $\pi_t = \kappa^{-1/\theta} Y_{t-1}/Y_t$.

17. Transversality?

So as usual, a Taylor-Principle Taylor rule implies there is a unique bounded path for the price level. But is there any equilibrium condition that is violated on the unbounded paths?

The private agent's problem has a concave objective function and its constraints (the budget constraint and a $B > 0$ constraint) are linear in decision variables and therefore define a convex constraint set. The one state variable for the private agent, B_t, is always "good", so the Lagrange multiplier on the constraint is always positive. And finally setting $B_t = 0$ is feasible. So the standard transversality condition, which here is $E[\beta^t B_t/(C_t P_t) \to 0$, applies. If this condition and the $B > 0$ constraint are satisfied on a path, the path is an equilibrium. But to assess whether these conditions are satisfied, we have to bring in the GBC and fiscal policy.

Substituting the fiscal policy rule into the GBC, dividing by Y_t, using the SRC to get rid of C_t, applying the E_{t-1} operator, and using the bond FOC, we get

$$E_{t-1} b_t^* = \beta^{-1} b_{t-1}^* + E_{t-1} \frac{\phi_0}{Y_t} - \phi_1 E_{t-1} b_t^* \,, \tag{18}$$

where $b_t^* = B_t/(P_t Y_t)$. We assume ε_{Ft} is i.i.d. across time, zero mean, and independent of Y_t. Then collecting terms in b_t^* and dividing through by the resulting coefficient gives us

$$E_{t-1} b_t^* = \frac{\beta^{-1}}{1+\phi_1} b_{t-1}^* + E_{t-1} \frac{\phi_0}{(1+\phi_1) Y_t} \,. \tag{19}$$

With an i.i.d. process for Y, the last term in this equation is a constant, so this is an ordinary, non-stochastic, linear difference equation in $E_{t-1} b_{t+s}^*$, $s = 0, \ldots, \infty$. With active fiscal policy ($\phi_1 \leq 0$), it is an unstable difference equation, so its only non-explosive solution is

$$b_t^* \equiv \bar{b} = \frac{1+\phi_1}{1+\phi_1-\beta^{-1}} E\left[\frac{\phi_0}{Y_t}\right] \,.$$

Thus we can conclude that if $\phi_1 \leq 0$ (and, necessarily then, $\phi_0 \leq 0$), $E[\beta^t b_t^*] \to 0$, and transversality is satisfied. This means we have satisfied sufficient conditions for a solution to the private agent's problem, and we have an equilibrium path. Note that this is true regardless of what kind of time path is followed by prices. If prices explode rapidly, the policy rule guarantees that the conventional (not primary) nominal deficit, $B_t - B_{t-1}$, itself explodes at the same rate, so that real debt remains stable despite the rapidly rising nominal debt.

Once we know the unique equilibrium value $\bar{b}$ of b_t^*, we can find the uniquely determined current price level from the original GBC (without the E_{t-1})

$$b_t^* = \bar{b} = R_{t-1}\frac{b_{t-1}^*}{\pi_t} - \phi_0 + \phi_1\bar{b} - \varepsilon_{Ft}\,.$$

In this equation everything is constant or predetermined at time $t-1$, except for π_t. So the equation uniquely determines π_t, and therefore P_t. Of course in general this π_t does not agree with the unique π_t that, with $\theta > 1$, delivers a stable price level. So our conclusion is that with active fiscal policy, there is a uniquely determined price level at every t, but if monetary policy is also active, it is unlikely that the path of inflation will be stable.

If instead $\phi_1 > \beta^{-1} - 1$ (passive fiscal policy), The difference equation (19) becomes stable. Then $E_t[b_{t+s}^*]$ converges to a constant as $s \to \infty$ no matter what the initial value of b^*. There is no uniquely stable value of b_t — every value of b_t is consistent with equilibrium. In other words, with passive fiscal policy and Taylor-Principle monetary policy, equilibrium is likely both to be explosive and to be non-unique.

The case of $0 < \phi_1 < \beta^{-1} - 1$ is actually a version of a passive-fiscal case. This case implies that there are explosive paths that grow at a rate slower than β^{-t}. On such paths, $E[\beta^t b_t^*] \to 0$, despite b^* growing arbitrarily large, so transversality is satisfied. In such equilibria, debt *and* primary surpluses τ grow without bound. Individuals do not feel tempted to spend down their wealth, even though it grows arbitrarily large relative to C_t, because they see themselves as needing the income from the ever-growing wealth to cover their current and future tax obligations, which are growing equally fast.

18. Passive money

If $\theta \leq 1$, we can no longer use (16) to determine a unique price level consistent with stability. However, our analysis above showed that the uniqueness of the initial price level is determined entirely by fiscal policy. When $\phi_1 > \beta^{-1} - 1$, there is a uniquely determined time path of prices, regardless of the value of θ. With $\theta > 1$, this time path, as we have shown, is likely to be explosive. With $\theta \leq 1$, the inflation rate will follow a path that is either stationary or (if $\theta = 1$) drifting, like a random walk. We will not verify this step by step here. It can be checked by substituting the monetary and fiscal policy rules for R_{t-1} in the GBC and replacing b_t and b_{t-1} with $\bar{b}$. This leads to a stochastic difference equation in π alone (and exogenous shocks). Taking logs, it can be seen to be stationary with $\theta < 1$, explosive with $\theta > 1$, and a random walk with drift for $\theta = 1$.

19. Exercise due Friday, 2/27

The US is now like most countries in that its central bank pays interest on reserves. To fully model such an economy, one would have to distinguish non-interest-bearing cash, interest-bearing reserves, and interest-bearing marketable government debt. As a first approximation, though, assume that all money pays interest at a rate somewhat below the rate on marketable debt, with money valuable for transactions reasons while debt is not.

We take the model from class and the "Simple model..." paper, in which agents solve

$$\max_{C,B,M} \sum_{s=0}^{\infty} \beta^t \log C_t \qquad \text{subject to} \tag{20}$$

$$C_t(1 + 2\gamma v_t) + \frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + R_{M,t-1}M_{t-1}}{P_t} - \tau_t + Y_t \tag{21}$$

$$B_t \geq 0\,, \qquad M_t \geq 0\,. \tag{22}$$

We assume monetary policy is $M_t \equiv \bar{M}$. We consider two kinds of interest rate differential policy:

$$R_{Mt} = R_t - \delta$$
$$R_{Mt} = 1 + \delta \cdot (R_t - 1)\,.$$

First, consider whether the price level can be determined uniquely from the FOC's plus the monetary policy rule alone, as is often true with an active money policy. Use the full non-linear specification. In ruling out unstable paths for velocity, you will want at some point to use the transversality condition for M. It will be enough for this exercise just to show that the TVC is violated to rule out a path, though you can get extra credit by making a direct argument that the paths cannot be optima. I believe you get different answers for the two interest rate differential policies.

If there is a case (my own calculations, which could be wrong, suggest there is) in which we cannot show a determinate price level without bringing in the GBC and fiscal policy, linearize the model around steady state and check whether the equilibrium is locally stable under active or passive fiscal policy. For this you will need to assume numerical values for parameters. Use $\beta = .9$, $\gamma = .01$. Use $\delta = .02$ for the first interest differential policy and $\delta = .8$ for the second. For the fiscal rule, try $\phi_0 = 1$, $\phi_1 = .15$ and $\phi_0 = -1$, $\phi_1 = 0$.