ANSWER FOR 2/27/09 EXERCISE

1. Exercise due Friday, 2/27

The US is now like most countries in that its central bank pays interest on reserves. To fully model such an economy, one would have to distinguish non-interest-bearing cash, interest-bearing reserves, and interest-bearing marketable government debt. As a first approximation, though, assume that all money pays interest at a rate somewhat below the rate on marketable debt, with money valuable for transactions reasons while debt is not.

We take the model from class and the "Simple model..." paper, in which agents solve

$$\max_{C,B,M} \sum_{s=0}^{\infty} \beta^t \log C_t \qquad \text{subject to} \tag{1}$$

$$C_t(1+2\gamma v_t) + \frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + R_{M,t-1}M_{t-1}}{P_t} - \tau_t + Y_t \tag{2}$$

$$B_t \geq 0\,, \qquad M_t \geq 0\,. \tag{3}$$

We assume monetary policy is $M_t \equiv \bar{M}$. We consider two kinds of interest rate differential policy:

$$R_{Mt} = R_t - \delta$$
$$R_{Mt} = 1 + \delta \cdot (R_t - 1)\,.$$

First, consider whether the price level can be determined uniquely from the FOC's plus the monetary policy rule alone, as is often true with an active money policy. Use the full non-linear specification. In ruling out unstable paths for velocity, you will want at some point to use the transversality condition for M. It will be enough for this exercise just to show that the TVC is violated to rule out a path, though you can get extra credit by making a direct argument that the paths cannot be optima. I believe you get different answers for the two interest rate differential policies.

It's a mystery to me why I put the "2" in the transactions cost term in the budget constraint, but it's there, so we'll proceed with it. The Euler equations for the private agent are

$$\begin{aligned}
\partial C: &\qquad \frac{1}{C_t} = \lambda_t \cdot (1 + 4\gamma v_t) \\
\partial B: &\qquad \frac{\lambda_t}{P_t} = \beta E_t \frac{R_t \lambda_{t+1}}{P_{t+1}} \\
\partial M: &\qquad \frac{\lambda_t}{P_t}(1 - 2\gamma v_t^2) = \beta E_t \frac{R_{Mt} \lambda_{t+1}}{P_{t+1}}
\end{aligned}$$

Taking the ratios of the B and M FOC's, we get

$$1 - 2\gamma v_t^2 = \frac{R_{Mt}}{R_t}\,.$$

The two policy rules then produce, respectively,

$$\begin{aligned}
1 - 2\gamma v_t^2 &= 1 - \frac{\delta}{R_t}\,, &\qquad \therefore R_t &= \frac{\delta}{2\gamma v_t^2} \\
1 - 2\gamma v_t^2 &= \frac{1-\delta}{R_t} + \delta &\qquad \therefore R_t &= \frac{1-\delta}{1 - 2\gamma v_t^2 - \delta}
\end{aligned}$$

Plugging these expressions back into the B FOC and multiplying the resulting equation by $\bar{M}$ gives us for the two policies the two difference equations in v:

$$\begin{aligned}
\frac{2\gamma v_t^2}{\delta v_t (1 + 4\gamma v_t)} &= \beta E_t \left[\frac{1}{v_{t+1}(1 + 4\gamma v_{t+1})} \right] \\
\frac{1 - 2\gamma v_t^2 - \delta}{(1-\delta) v_t (1 + 4\gamma v_t)} &= \beta E_t \left[\frac{1}{v_{t+1}(1 + 4\gamma v_{t+1})} \right]\,.
\end{aligned}$$

If we let $Z_t = 1/(v_t(1 + f\gamma v_t)$, we see that Z_t is monotone decreasing in v_t, with $v_t \to 0$ as $Z_t \to \infty$ and $v_t \to \infty$ as $Z_t \to 0$. The two difference equations can again be rewritten as

$$\begin{aligned}
\frac{2\gamma v_t^2}{\delta} Z_t &= \beta E_t Z_{t+1} \\
\frac{1 - 2\gamma v_t^2 - \delta}{1-\delta} Z_t &= \beta E_t Z_{t+1}\,.
\end{aligned}$$

Both of these equations have as one solution a deterministic steady state. The coefficient on Z_t on the left-hand side can in each case match any $\beta \in (0, 1)$ at some positive value of v_t, and this value of v_t, and the corresponding interest rate, can then persist forever as a constant solution to the difference equation. However the coefficient changes in opposite directions as a function of v_t in the two cases. The upper equation makes the coefficient on Z_t increase in v_t, while the lower one makes it decrease. So if Z moves above its steady state value in the upper-equation case, and therefore v moves below its steady state value, the coefficient on the left-hand side drops below one, tending to make the next value of Z lower. The equation is therefore at least locally stable and no argument based on showing unstable paths are impossible will work to prove that the constant-v solution is the only one.

In the lower equation, a rise in Z above steady state raises the coefficient on lagged Z above one. The equation is therefore locally unstable, and an argument like that we gave in class, in the notes, and in the "Simple Model..." paper shows that once Z goes above its steady state value it has a non-zero probability of going above any given upper bound. A similar argument shows that once it goes below the steady state value, it has a non-zero probability of going arbitrarily close to zero. Z arbitrarily close to zero, though, implies v arbitrarily large, and one can see that the left hand side of the lower equation eventually would become negative for large v, making it impossible to satisfy the equation at any positive (and hence at any feasible) value of v. So equilibria where Z shrinks to zero can be ruled out. If Z grows arbitrarily large, v must approach zero. Assuming Y_t is bounded away from zero, C_t is bounded away from zero for small values of v, and $M_t = \bar{M}$ is fixed, so to have $v_t \to 0$ we must have $P_t \to 0$ and real balances $\bar{M}/P_t \to \infty$. Since the problem states that both M and B must remain non-negative, we can apply the transversality condition separately to the two. (Otherwise transversality would apply only to their sum, and the two could go off to infinity and minus infinity so long as their sum did not explode.) The transversality condition for M is

$$E\left[\beta^t \frac{M_t(1-2\gamma v_t^2)}{P_t C_t(1+4\gamma v_t)}\right] = E[\beta^t(1-2\gamma v_t^2)Z_t] \to 0\,.$$

The difference equation for Z_t makes it clear that $E_t Z_{t+1} < \beta^{-1} Z_t$, for every t, but as $v_t \to 0$ the expected rate of growth approaches β^{-1} from below. It is therefore a bit tricky to verify rigorously that $\beta^t E Z_t$ fails to go to zero on these paths. However a direct argument that such paths cannot be optimal is easier. Suppose on such a path the agent reduces real money balances at date t by the amount x. That is, instead of the original solution path $M_t = \bar{M}$, he chooses to make $M_s = \bar{M} - x$ for $s > t$. He converts this money into consumption at t, and leaves plans for B_s for $s > t$ unchanged. This is feasible. It leaves him spending exactly the same amount on consumption inclusive of transactions costs at later dates as on the original path. Clearly he gets a utility boost at date t. He will have lower C, and thus lower utility, at later dates, however, because the lower M/P will increase transactions costs. However, because $C_s(1+\gamma v_s)$ remains unchanged at all dates later than t, the effect on discounted utility is only the effect on

$$\sum_{s=0}^{\infty} \beta^s \log(1 + 2\gamma v_{t+s})\,.$$

Since $m = M/P$ has been decreased by the same proportional amount $(\bar{M} - x)/(\bar{M})$ at all dates, the effect on C is bounded. In fact, one can show that as $v \to 0$, $d\log C/d\log M \to -.5$. Thus the future utility cost of a given percentage reduction in M at t converges to a finite limit as the value of m_t increases. But the current utility gain from consuming a given percentage of M at t grows arbitrarily large as $m \to \infty$. Thus no solution path for the Z equation on which m_t is unbounded above can be optimal.

So we have shown that for the policy that makes the money net interest rate lower than that for debt by a fixed proportion there is a unique stable solution, without looking at the budget constraint or fiscal policy. Of course this assumes implicitly that fiscal policy will keep the debt stable, so as not to violate the B TVC.

If there is a case (my own calculations, which could be wrong, suggest there is) in which we cannot show a determinate price level without bringing in the GBC and fiscal policy, linearize the model around steady state and check whether the equilibrium is locally stable under active or passive fiscal policy. For this you will need to assume numerical values for parameters. Use $\beta = .9$, $\gamma = .01$. Use $\delta = .02$ for the first interest differential policy and $\delta = .8$ or the second. For the fiscal rule, try $\phi_0 = 1$, $\phi_1 = .15$ and $\phi_0 = -1$, $\phi_1 = 0$.

So there is such a case and we need to linearize and check.

The first interest differential policy rule, which makes the differential between debt and money interest rates a constant, behaves like a passive monetary policy, while the second behaves like an active rule, according to our analysis above of the Z difference equation. So is is not surprising that analysis with `gensys` tells us that we have existence and uniqueness for the absolute differential rule when $\phi_0 = -1$, $\phi_1 = 0$, but non-uniqueness when $\phi_0 = 1$, $\phi_1 = .15$. This fits perfectly Leeper's (passive money, active fiscal) and (passive money, passive fiscal) cases. The second rule produces existence and uniqueness under $\phi_0 = 1$, $\phi_1 = .15$, but non-existence when $\phi_0 = -1$, $\phi_1 = 0$, again fitting Leeper's classifications.

An R program that calculates the model's steady states and generates the inputs for `gensys` from given parameter values is attached. Note that using R's ability to label matrix rows and columns and to address matrices with character row and column indexes makes the program easier to understand, write, and debug. Note also that I preserved the variables Z and v, even though they could have been substituted out, making the system 5×5 instead of 7×7. The advantage of sticking to the larger system is that it makes understanding and checking the equations easier. If we were worried mainly about numerical efficiency, reducing the system size might have made sense.

The output of this program is used by first setting the parameter values (including the string variable `policy`) and then running lines like those below.

```
> exout <- Ex227Ans(gam, bet, delt, phi0, phi1, ybar, policy)
> gout <- with(exout, gensys(g0, g1, psi=Psi, pi=Pi))|
```

```
Ex227Ans <- function(gam, bet, delt, phi0, phi1, ybar, policy=c("abs","rel")) {
  absdel <- policy=="abs"
  Rbar <- 1/bet
  vbar <- if (absdel) sqrt(delt*bet/(2*gam)) else sqrt((1 - delt) * (1 - bet)/(2 * gam))
  cbar <- ybar/(1 + 2 * gam * vbar)
  mbar <- cbar/vbar
  pibar <- 1
  bbar <- if (absdel) {
    ((1 - 1/bet - delt) * mbar - phi0)/(1/bet -1 - phi1)
  } else {
    ( -(1/bet - 1) * (1 - delt) * mbar - phi0) / (1/bet -1 - phi1)
  }
  ss <- c(Rbar=Rbar, vbar=vbar, cbar=cbar, mbar=mbar, pibar=pibar, bbar=bbar)
  g0 <- matrix(0,7,7)
  g1 <- g0
  Psi <- matrix(0,7,1)
  Pi <- matrix(0,7,1)
  dimnames(g0) <- list(eq=c("zde","liqp", "gbc", "src", "vdef", "zdef", "Mconst"),
                       var=c("z", "R", "v", "b", "m", "pi", "c"))
  dimnames(g1) <- dimnames(g0)
  g0[1, "z"] <- 1
  g0[1, "R"] <- 1
  g1[1, "z"] <- 1
  Pi[1, 1] <- 1
  g0[2, "v"] <- 4 * gam * vbar^2
  g0[2, "R"] <- ifelse(absdel, delt * bet,  -bet * (1 - delt))
  g0[3, "b"] <- (1 + phi1)
  g0[3, "m"] <- 1
  g0[3, "pi"] <- (bbar + mbar) / bet - ifelse(absdel, delt * mbar, delt * (1/bet -1) * mbar)
  g1[3, "b"] <- 1/bet
  g1[3, "m"] <- 1/bet - ifelse(absdel, delt, delt / bet)
  g1[3, "R"] <- (bbar + mbar)/bet - ifelse(absdel, 0, delt * mbar / bet)
  g0[4, "c"] <- 1
  g0[4, "v"] <- 2 * gam * vbar/(1 + 2 * gam * vbar)
  Psi[4, 1] <- 1
  g0[5, "v"] <- 1
  g0[5, "c"] <- -1
  g0[5, "m"] <- 1
  g0[6, "z"] <- 1
  g0[6, "v"] <- 1 + 4 * gam * vbar / (1 + 4 * gam * vbar)
  g0[7, "v"] <- 1
  g0[7, "pi"] <- -1
  g0[7, "c"] <- -1
  g1[7, "v"] <- 1
  g1[7, "c"] <- -1
  return( list( ss=ss, g0=g0, g1=g1, Psi=Psi, Pi=Pi))
}
```