Exercise on Lagrange Multipliers and Transversality

This is a stylized model of a world reliant on nuclear power. There is a finite stock R of "uranium", some of which can be "burned" (B) each period to produce a consumption good that yields utility. The technology for turning burnt fuel into consumption goods is characterized by the production function $f(B_t, \varepsilon_t)$, where ε_t is an exogenous stochastic process. Burning uranium also, however, adds to the stock of "waste" (W), which decreases utility. The problem is how to choose the optimal path of immiseration in this world.

The representative agent has utility function $U(C_t, W_t)$, with $D_C U(C_t, W_t) > 0$, $D_W(C_t, W_t) < 0$ and U concave. The planner's problem is

$$\max_{C,W,R,B} E\left[\sum_{t=1}^{\infty} \beta^t U(C_t, W_t)\right] \text{ s.t.} \tag{1}$$

$$C_t \leq f(B_t, \varepsilon_t) \tag{2}$$

$$R_t \leq R_{t-1} - B_t \tag{3}$$

$$W_t \geq (1-\delta)W_{t-1} + B_t \tag{4}$$

$$R_t \geq 0 \qquad B_t \geq 0 \tag{5}$$

$$R_0 > 0 \text{ given;} \qquad W_0 = 0 \text{ given.} \tag{6}$$

(a) If U and f are concave, verify that this problem satisfies the conditions of the sufficiency theorem for the general TVC laid out in class. (You may have to redefine some variables to flip their signs.)

This question was not worded carefully, and I did not give enough assumptions on f and U to allow a complete answer without invoking additional assumptions. What I had in mind was the convexity and concavity assumptions. The W dynamics constraint and the two positivity constraints are written with "$\geq$", so they have to be flipped, or have their signs reversed, in order to match the canonical form in the theorem. But since they are linear, those constraints are clearly convex. The only nonlinear constraint, that relating C to B, is convex so long as f is concave. But of course one also needs differentiability of U and f and boundedness of the expectation of the derivatives, which require further assumptions on U and f

(b) Find the Euler equations and the TVC for the problem.

Euler equations are:

$$\begin{aligned}
\partial C: &\qquad D_C U_t = \lambda_t \\
\partial W: &\qquad D_w U_t = -\nu_t + (1-\delta)\beta E_t \nu_{t+1} \\
\partial R: &\qquad \mu_t = \beta E_t \mu_{t+1} + \theta_t^R \\
\partial B: &\qquad \lambda_t f'(B_t, \varepsilon_t) = \beta E_t \mu_{t+1} + \nu_t - \theta_t^B .
\end{aligned}$$

Here λ, μ, ν, θ^R, and θ^B are the respective Lagrange multipliers on the constraints above, in the order in which the constraints are listed. The conditions allowing the "standard" TVC are not quite met here. It is natural to suppose that R and B

could always feasibly be set to zero. But the same is not true of W, because B has to be non-negative. So we need to look at the more general form of the TVC. It requires that for any feasible ΔR, ΔW, and ΔB sequences,

$$\liminf_{t\to\infty} \beta^t E[((-\mu_t + \theta_t^R)\Delta R_t + (D_w U_t + \nu_t)\Delta W_t + (\lambda_t f_t' + \theta_t^B - \nu_t)\Delta B_t] \le 0 \,.$$

Note that from the ∂R Euler equation we know that if the second constraint is binding (which it clearly will be) and $R > 0$ (so $\theta^R = 0$), μ must be growing in expectation, at the rate β^{-t}. Also, because R can only decrease and is bounded below, R_t must converge to a limit. If the limit is positive, then a deviation from it down to $R_t = 0$ (a negative ΔR_t) could be sustained for all t. This would make $\beta^t E[\mu_t]\Delta R_t$ converge to a finite, positive limit. This would require deviations in B and W as well. If those deviations did not produce offsetting, negative components in the TVC, this first component of the TVC is requiring that all the uranium eventually be used up. The second component requires that waste, weighted by its marginal disutility, not be accumulated too rapidly. The third rules out paths in which the shadow value of reduced waste grows rapidly and at the same time B_t does not shrink rapidly enough.

(c) Suppose $f(B_t, \varepsilon_t) = \alpha B_t + \varepsilon_t$, $U(C_t, W_t) = C_t - \frac{1}{2}C_t^2 - \frac{1}{2}W_t^2$, Assume ε is i.i.d. with mean 0, $\alpha > 0$, $\delta \in (0,1)$. See if you can obtain an explicit solution for the optimal path. [Solving this may or may not be feasible. The problem is LQ if all constraints bind either always or never on the solution path. If R ends up converging exponentially to a non-negative constant, this might work. But if $R \ge 0$ necessarily binds at some point, the problem will be messy and non-LQ. Note that the quadratic utility function is well defined for negative C, and we are not directly ruling out negative C, but the $B \ge 0$ constraint implies $E[C_t] \ge 0$. You are either to explicitly solve, or explain why the solution is necessarily hard.]

These specific functional forms do not simplify things much. The R FOC tells us that if the $R > 0$ constraint never binds, expected μ must grow at the rate β^{-t}. But the FOC's can be used to derive the condition $\lambda_t = \mu_t + \nu_t$. All the Lagrange multipliers must remain non-negative in a solution, so the exponential growth in μ requires unboundedness of λ. But with this utility function, $\lambda_t = 1 - C_t$ and is therefore bounded between 0 and 1. Thus any optimal solution must eventually make the $R \ge 0$ constraint bind. Unless the solution turns out to be $B_1 = R_0$, so $C_t = 0$ for $t > 1$, the solution will involve a switch from non-binding to binding $R_t \ge 0$, and thus certainty equivalence will not apply, and the solution will be hard.