## EXERCISE ON OPTIMAL FISCAL POLICY

This exercise considers a model much like that in Lucas and Stokey (1983). They assume debt instruments of arbitrary maturity with state-contingent returns and that the government can commit to arbitrary future streams of state-contingent rates of return on debt, but not to future tax rates. This exercise assumes one-period state-contingent debt and that the government has no commitment problem. Both setups imply that the fully optimal solution with no constraint requiring debt to be non-negative would be an initial tax to create negative debt large enough that no distorting taxation is ever necessary, and both setups assume that this for some reason does not occur. The model differs from what we have studied in class in that it studies real debt, not nominal debt, and allows the return on debt issued at $t$ to be a random variable whose value is determined only at $t+1$.

We assume that the economy is made up of identical representative agents, each of whom solves

$$\max_{C,L,B} E\left[\sum_{t=0}^{\infty}(\log C_t + 1 - L^2)\beta^t\right] \quad \text{subject to} \tag{1}$$

$$\lambda: \quad C_t + B_t + \tau_t L_t \leq R_t B_{t-1} + 2L_t - L_t^2 \tag{2}$$

$$\psi_1: \quad B_t \geq 0 \tag{3}$$

$$\psi_{2,3}: \quad 0 \leq L_t \leq 1\,, \tag{4}$$

taking $\tau$ and $R$ to be exogenous stochastic processes (i.e. unaffected by the agent's own choices). The Greek letters at the left of constraints, here and below, are Lagrange multipliers for those constraints that will appear in first order conditions (FOC's). The FOC's for the private agents' problem, assuming the constraints on the ranges of $B$ and $L$ are not binding (so the $\psi$'s are all zero) are

$$\mu_1: \quad \frac{1}{C_t} = \lambda_t \tag{5}$$

$$\mu_2: \quad -2L_t = \lambda_t(\tau_t - (2 - 2L_t)) \tag{6}$$

$$\mu_3: \quad \lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}] \tag{7}$$

The government wishes to maximize the welfare of the representative agent, recognizing that the labor tax rate $\tau_t$ distorts labor-leisure choices and that the timing of taxes can be adjusted by use of debt. The government therefore maximizes (1) subject to the agents' budget constraint, the agents' FOC's, and, in addition, to its own budget constraint and range constraints. It is convenient to subtract the government's budget constraint from the representative agents' constraint to obtain a social resource constraint, so that the government's problem is to maximize (1) subject to the private FOC's above, the private range constraints, plus

$$\phi: \quad B_t + \tau_t L_t = R_t B_{t-1} + g_t \tag{8}$$
$$\nu: \quad C_t + g_t = 2L_t - L_t^2 \tag{9}$$
$$\xi: \quad \tau_t \leq 1\,. \tag{10}$$

The government's choice variables consist of all the private sector choice variables, including $\lambda$, plus $R$ and $\tau$. As usual in this course, variables dated $t$ are known at $t$ and/or chosen at $t$, meaning that the government chooses $R_t$, the return on time-$(t-1)$ bonds, at time $t$. The government does not choose expenditures $g_t$, which is taken as an exogenously given stochastic process.

(i) Assuming (to limit the complexity of the algebra) that none of the range constraints are binding (so that the $\psi$'s and $\xi$ are zero), derive the FOC's for the government's problem. Explain how the time-zero constraints differ from those after time zero and how this reflects the time-consistency problem.

Here are the government's FOC's:

$$\partial C: \quad \frac{1}{C_t} = -\frac{\mu_{1t}}{C_t^2} + \nu_t$$
$$\partial L: \quad -2L_t = -2(1+\tau_t)\mu_{2t} + \phi_t\tau_t - 2\phi_t(1-L_t)$$
$$\partial B: \quad 0 = \phi_t + \beta E_t[R_{t+1}\phi_{t+1}]$$
$$\partial R: \quad 0 = -\mu_{3,t-1}\lambda_t - \phi_t B_{t-1}$$
$$\partial \lambda: \quad 0 = -\mu_{1t} - \mu_{2t}(\tau_t - (2-2L_t)) + \mu_{3t} - \mu_{3,t-1}R_t$$
$$\partial \tau: \quad 0 = -\mu_{2t} + \phi_t L_t$$

At time 0, the time t=-1 version of (7) is not part of the problem, so at time 0 all the occurrences of $\mu_3$ are zero, and in particular the $\partial\lambda$ and the $\partial R$ constraints are of a different form. This reflects the fact that at time zero the expectations of the public at time -1 are not a constraint on policy behavior, whereas (under commitment) they are a constraint at later dates.

(ii) Show that these FOC's imply that $\phi_t/\lambda_t$ (the ratio of the government's Lagrange multiplier on its budget constraint to the private agents' multiplier on their budget constraint) is constant after the initial period. Below we refer to this constant as $\gamma$.

Substituting the $\partial R$ constraint into the $\partial B$ constraint leads to

$$\frac{\mu_{3,t-1}\lambda_t}{B_{t-1}} = \beta E_t\left[\frac{R_{t+1}\lambda_{t+1}\mu_{3t}}{B_t}\right]\,.$$

Then using the private $\partial C$ FOC (5), we get

$$\frac{\mu_{3,t-1}}{B_{t-1}} = \frac{\mu_{3t}}{B_t}\,,$$

which then from the $\partial R$ FOC implies that $\phi_t/\lambda_t$ is constant.

(iii) Derive a system equations that can be solved for $\tau_t$ as a function of $\gamma$ and $g_t$ alone.

From the $\partial\tau$ FOC and the fact that $\phi_t = \lambda_t\gamma = \gamma/C_t$ we get $\mu_{2t} = \gamma L_t/C_t$. From the previous question we get that $\mu_{3t} = -B_t\gamma$. From the GBC, we know that $B_t - R_t B_{t-1} = g_t - \tau_t L_t$. Using all these facts in the $\partial\lambda$ FOC, we arrive at

$$-\mu_1 t - \frac{\gamma L_t}{C_t}(\tau_t - (2 - 2L_t)) + \gamma(\tau_t L_t - g_t)\,.$$

Similar, but simpler substitutions in the $\partial L$, and $\partial C$ FOC's allow us to eliminate occurences of $\phi$ and $\mu_2$ from those equations. Then these two equations, the rewritten $\partial\lambda$ FOC above, and the social resource constraint (with $\lambda$ substituted out) are a system of four equations in the five variables $\mu_{1t}$, $C_t$, $L_t$, $\tau_t$, and $g_t$. We should be able to solve the system to express the other four variables as functions of $g_t$.

(iv) Show that if $g_t$ is i.i.d., $B_t/C_t$ is constant after the initial period.

Dividing the GBC through by $C_t$ gives us

$$\frac{B_t}{C_t} = \frac{R_t B_{t-1}}{C_{t-1}} \cdot \frac{C_{t-1}}{C_t} + \frac{g_t - \tau_t L_t}{C_t}\,.$$

Using a) the fact that $g_t$, $\tau_t$, $C_t$ and $L_t$ are all functions of $g_t$ and hence i.i.d. and b) the consumer's $\partial B$ FOC, we can take $E_{t-1}$ of the equation above to arrive at

$$E_{t-1}\left[\frac{B_t}{C_t}\right] = \beta^{-1}\frac{B_{t-1}}{C_{t-1}} + \sigma\,,$$

where $\sigma = E[(g_t - \tau_t L_t)/C_t]$. This is an unstable difference equation for $\beta \in (-1, 1)$. The TVC here is $E[\beta^t B_t/C_t] \to 0$, so the only solution consistent with equilibrium is the unique stable solution, $B_t/C_t \equiv \sigma/(1-\beta^{-1})$.

(v) Show that, therefore, once it has set the optimal tax rate $\tau_t$, the government has no remaining freedom to choose $R_t$ – there is only one $R_t$ consistent with the amount of current debt $B_t$ that the public is willing to hold.

Since the left-hand-side of (iv) is fixed at $\sigma/(\beta^{-1}-1)$, and the right-hand side variables are, except for $R_t$, all either determined by $g_t$ or predetermined, $R_t$ must take on the unique value consistent with equality of the left and right hand sides. Putting the same point another way, the government really only has one degree of freedom in setting policy in this environment, where there is no price level and hence no monetary policy. Once it has set $\tau_t$, it has in effect also set $R_t$. Putting the same point yet another way, the government's vector of choice variables is six-dimensional (as we can see from its six FOC's). It faces five constraints — the GBC, the SRC, and the three private FOC's. Thus there is only one dimension of free variation.

(vi) Could the real allocation in this equilibrium be achieved by a government that issued nominal, rather than real debt, set $\tau_t$ as a function of $g_t$ just as in this real-debt optimal policy equilibrium, fixed the nominal interest rate at some constant, and then allowed the price level to adjust to generate the random return on debt?

We haven't actually solved the model above, just done counting of equations and variables. Assuming the equations do have a solution, it does seem that the policy could be implemented by setting the $\tau_t$ variable to the values from the optimal solution, fixing the nominal interest rate, and letting $P_t$ adjust. The argument that, once $\tau_t$ is set, there is only one $R_t$ consistent with the GBC, would apply as before, but now with $R_t = \bar{R}P_{t-1}/P_t$. There would be a unique $P_t$ consistent with equilibrium, and it would be set by the market supply and demand for nominal government debt at the fixed nominal interest rate $\bar{R}$.

## References

Lucas, Robert E., J., and N. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12(1), 55–93.