FINAL EXAM

(1) (45 points) Consider a variant of Leeper's setup in which we introduce demand for money. Private sector behavior is described by

$$\text{Fisher equation}: \quad r_t = \rho + E_t \pi_{t+1} \tag{1}$$

$$\text{Money demand}: \quad m_t = \bar{v} + p_t + y_t - \theta r_t + \nu_t \tag{2}$$

$$\text{Exogenous } y: \quad y_t - \bar{y} = \gamma(y_{t-1} - \bar{y}) + \varepsilon_t\,. \tag{3}$$

Here r is the interest rate, m, p, and y are logs of money, price level, and output, respectively, and $\pi_t = p_t - p_{t-1}$ is the inflation rate. Assume that fiscal policy is passive, so that for any price level path and with interest rates satisfying the Fisher equation, real debt is stable.

(a) Suppose monetary policy fixes the money stock, i.e. sets $m \equiv \bar{m}$. Is there a unique equilibrium price stochastic process? Does your conclusion depend on the values of θ and γ? What are the signs of the effects of output shocks ε_t and money demand shocks ν_t on the price level?

The simplest way to solve this problem, is to reduce it to a one-dimensional difference equation. Solving the money demand equation for r_t and equating the result to the right-hand-side of the Fisher equation produces an equation in p_t, y_t and constants alone, which can be written as

$$E_t p_{t+1} = \frac{1+\theta}{\theta} p_t + \frac{1}{\theta} y_t + \frac{\bar{v} - \bar{m}}{\theta} + \frac{\nu_t}{\theta} - \rho\,.$$

In order for this to be solvable forward only, and thus to deliver a unique stable solution on the assumption that y_t is stationary, we require $\frac{1+\theta}{\theta} > 1$, but this is true for any $\theta > 0$, i.e. whenever the demand for real balances responds negatively to the nominal interest rate. In that case, the forward solution is

$$p_t = E_t \left[\sum_{t=0}^{\infty} \left(\frac{\theta}{1+\theta} \right)^s \frac{1}{\theta} (-y_t - \bar{v} + \bar{m} - \nu_{t+s} + \theta\rho) \right]$$

The problem failed to make it explicit that ε_t and ν_t were serially independent and zero mean, though it did characterize them as "shocks". Assuming that they are serially independent and zero mean, we have $E_t y_{t+s} - \bar{y} = \gamma^s (y_t - \bar{y})$ and $E_t \nu_{t+s} = 0$ for $s > 0$. This makes the solution emerge as

$$p_t = \frac{-y_t(1+\theta)}{1+(1-\gamma)\theta} + \bar{y} - \bar{v} + \bar{m} - \frac{\nu_t}{1+\theta} + \rho\theta\,.$$

From this it is easy to see that the effect of output shocks and money demand shocks on the price level are both negative.

A really good answer would have noted that the signs are indeterminate if we know nothing about the serial correlation properties of ε_t and ν_t. It would also have noted that the equations of the model as given do not describe the optimization problems of individuals in the economy or any borrowing constraints on government that might be relevant, so that in excluding explosive solutions we are simply assuming that transversality or feasibility conditions that have not been specified would eliminate these paths as potential equilibria.

(b) Suppose monetary policy doesn't absolutely fix the money stock, but instead raises interest rates when money increases above a target level, i.e. sets $r_t = \rho + \alpha(m_t - \bar{m})$. Is there a unique equilibrium price stochastic process? Does your conclusion depend on the values of θ, γ, and α? What are the signs of the effects of ε_t and ν_t in this case?

This can be answered by almost the same algebraic maneuvers as the previous part. The resulting first order single-endogenous-variable equation is

$$E_t p_{t+1} = \frac{1 + \alpha(\theta + 1)}{1 + \theta\alpha} p_t + \frac{\alpha}{1 + \alpha\theta} (y_t + \nu_t + \bar{v} - \bar{m} - \theta\rho) \, .$$

This is unstable so long as the coefficient on p_t exceeds one, which will be true for any $\alpha > 0$, $\theta > 0$ pair, As in the previous part, a full solution shows that both ε_t and ν_t have negative effects on p_t in the stable equilibrium. [This second part added algebra without adding much substance. I originally had the policy rule responding to the growth in m, $m_t - m_{t-1}$, rather than to $m_t - \bar{m}$. This leads to a second-order difference equation whose full solution seemed to me perhaps too much to ask on a 45 minute question. If I'd had more time to correct this, I would have preserved the second-order dynamics and asked for a solution in guided steps. This information is useful mainly to people who study this exam while preparing for subsequent-year versions..]

[Common errors: Quite a few people found the single-variable difference equation and noted the condition on its root needed to imply a unique stable equilibrium, but did not finish the forward solution before trying to draw conclusions about the signs of effects of ε_t and ν_t shocks. The full solution depends on expected future values of y_t and ν_t, not just the current values, so to find the signs of the effects one has to first show how current p_t depends on expected future exogenous variables, then discuss how those expectations depend on ε_t and ν_t. Several people cast the problem into matrix form, as you had to do on one homework, but then proceeded no further, not even mentioning that casting the problem in this form is useful mainly to allow its being solved by computer.]

(2) (45 points) Consider a simple overlapping generations model with a capital tax. Population is constant, and every individual in the generation born at t (which we call "generation t" henceforth) lives from t to $t+1$ and maximizes

$$\log(C_{1t}) + \log(C_{2,t+1}) \tag{4}$$

subject to

$$C_{1t} + Q_t K_{1t} + K_{2t} = AK_{1t} + Y + g_t \tag{5}$$

$$C_{2,t+1} = (1-\tau_{t+1})Q_{t+1}K_{2t} \tag{$*$}$$

$$K_{2t} \geq 0 \tag{6}$$

$$K_{1t} \geq 0, \tag{7}$$

where C_{1t} is consumption at time t of the generation born at t, $C_{2,t+1}$ is consumption at time $t+1$ of the generation born at t, K_{1t} is the capital purchased at time t by generation t from generation $t-1$, and $K_{2,t}$ is the capital saved up by generation t to be sold at $t+1$ to finance retirement. τ_t is the rate of capital taxation at t and g_t is lump sum transfers by the government to the younger generation at t. The government budget constraint is $g_t = \tau_t Q_t K_{2,t-1}$, and market clearing requires $K_{1,t} = K_{2,t-1}$. We think of the economy as starting at $t=0$, at which time there is an older generation that has capital to sell in the amount $K_{2,-1}$, which we take as given.

(a) Assuming τ is constant, find the steady state competitive equilibrium values of C_1 and C_2 as functions of A, Y, and τ.

The FOC's of agents born at t are

$$\begin{aligned}
\partial C_{1t}: &\quad \frac{1}{C_{1t}} = \lambda_t \\
\partial C_{2,t+1}: &\quad \frac{1}{C_{2,t+1}} = \mu_{t+1} \\
\partial K_1 t: &\quad Q_t = A \\
\partial K_{2t}: &\quad \lambda_t = (1-\tau_{t+1})Q_{t+1}\mu_{t+1}
\end{aligned}$$

where λ_t and μ_{t+1} are the Lagrange multipliers on first and second period budget constraints and we have assumed that the solution will not involve binding $K_{1t} > 0$ or $K_{2,t+1} > 0$ constraints. The social resource constraint, obtained by adding the first period and a lagged second period constraint, is

$$C_{1t} + C_{2t} + K_{2t} = AK_{1t} + Y\,.$$

Substituting for Q_t, λ_t and μ_t using the first three FOC's, the third FOC becomes

$$\frac{C_{2,t+1}}{C_{1t}} = (1-\tau_{t+1})A\,.$$

With log utility, the fraction of wealth saved generally is insensitive to rates of return. In this model, that turns out to be true. From the expression above, together with the second period budget constraint $(*)$, we get

$$C_{2,t+1} = (1-\tau_{t+1})AC_{1t} = (1-\tau_{t+1})AK_{2t}\ ,$$

from which it is easy to see that we must have $C_{1t} = K_{2t}$. In other words, regardless of the tax rate (or A, for that matter) the amount consumed matches the amount saved in the first period of life for each generation. This lets us find

$$C_{1t} = K_{2t} = \frac{Y_t + \tau_t AK_{2,t-1}}{2} \qquad (\dagger)$$
$$C_{2,t+1} = (1-\tau_{t+1})A\frac{Y_t + \tau_t AK_{2,t-1}}{2}\ .$$

In steady state $K_t = K_{2t}$ and all variables are constant over time. Using this in the above expressions lets us calculate the steady state values (where we drop the t subscripts)

$$K = \tfrac{1}{2}\frac{Y}{1-\frac{1}{2}\tau A}$$
$$C_1 = \tfrac{1}{2}\frac{Y}{1-\frac{1}{2}\tau A}$$
$$C_2 = \tfrac{1}{2}(1-\tau)A\frac{Y}{1-\frac{1}{2}\tau A}\ .$$

Note that the solution only works for $\tau < 2/A$, since for τ above that the difference equation in K_2 $(\dagger)$ is unstable. Of course we also need $\tau < 1$ to avoid zero consumption in the second period, so the unstable solutions only exist for $A > 2$. With $A < 2$, an increase in τ increases steady state C_1 and decreases steady state C_2.

(b) Show that the derivative of agent utility with respect to τ in steady state is negative at $\tau = 0$ under reasonable assumptions (say what they are) on A.

Unfortunately, what I asked you to prove here is true only when $A < 1$. I had checked this myself, but made the same algebra mistake twice when I was preparing the exam. The derivative of $\log C_1 + \log C_2$ with respect to τ is

$$\frac{A}{1-\frac{1}{2}\tau A} - \frac{1}{1-\tau}\ .$$

Evaluated at $\tau = 0$, this expression is $A - 1$ — positive for $A > 1$, negative for $A < 1$.

[Why does a capital tax raise steady-state welfare here? It's not really the capital tax that's doing the work. It's the transfer from the old to the young. This makes people poorer in old age, richer when young, and thereby encourages saving, even though the rate of return has been lowered. The usual result that steady state capital gets driven down by a capital tax does not apply here, because that effect works off the notion that the marginal product of capital adjusts to match the discount rate. Here the marginal product of capital is fixed. With $A > 1$ a planner could drive steady state utility arbitrarily high by enforcing low C_1 for a long time, thereby making K arbitrarily large. This could not generally be achieved by increasing τ, however, because unless $A > 2$, every feasible τ just leads to a fixed steady state K. If $A > 2$, a high tax rate will set the economy on an expansion path in which K, and hence C_1 and C_2, grow arbitrarily large. With $A < 1$ the competitive no-tax equilibrium exhibits dynamic inefficiency. The less people save, the more resources are available. So a positive tax and transfer, by increasing saving, worsen the inefficiency. A capital subsidy ($\tau < 0$) increases welfare by reducing saving, but it cannot achieve the first-best solution; any finite negative τ still leads to a steady state $K > 0$, and as $\tau \to -\infty$, C_1 goes to zero along with K, which is far from optimal.]

(c) We argued in class that in a single-agent model the derivative of agent utility with respect to a capital tax was zero at $\tau = 0$, because initial gains were offset by discounted future losses. Is there an analogous result in this model? [Hint: Consider first the agents born at $t = 0$ and the initial old.] The tax is on sales of capital by the old, so the initial old certainly lose from imposition of the tax. They simply pay for a transfer to the young. In the no-tax case, K_{2t} goes immediately to the no-tax steady state value of $Y/2$ and stays there in subsequent generations. With the tax, it instead converges exponentially to its steady state value (outside the cases mentioned above where there is no steady state). Since the steady state K is larger the larger is τ (when $A < 2$), All generations after the first benefit. If the initial old have the no-tax steady state amount of capital to sell, then each generation after the first benefits more, as the transfer they receive steadily rises and the rate of return stays constant. These effects are all first order. The only analogue to the single-agent result would arise if we model the no-tax equilibrium as reflecting a planner's solution that uses a social welfare function weighting individuals' utilities, discounting them at the rate $\beta = A^{-1}$. Then the discounted present value of the future utility gains matches the utility loss of the initial old. However, a planner discounting generations at any rate other than A^{-1} would never choose to generate a steady state. [Common error: Quite a few people substituted the government budget constraint into the individual's budget constraint before taking FOC's. That constraint, which makes the transfer received by the young depend on the previous generation's K_2 or (equivalently) the current generation's K_1, is a constraint on the government, not individuals. The problem stated that individuals see g_t as a lump sum transfer — that is, they do not see the size of the transfer as depending on any variable they choose.]