TRANSVERSALITY AND HABIT FORMATION EXERCISE

When trying to match the monthly or quarterly data, economists often find they need to modify the standard permanent income model to allow "habit formation" — a preference of consumers for not changing consumption rapidly. Here is a model that does so:

$$\max_{C,W} \sum_{t=0}^{\infty} \beta^t (C_t^\theta + C_{t-1}^\theta)^{1/\theta} \text{ subject to} \tag{1}$$

$$W_t \leq R(W_{t-1} - C_{t-1}) - \bar{Y} \tag{2}$$

$$0 \leq W_t \,. \tag{3}$$

Notice that we are not introducing any uncertainty here. Income, $\bar{Y}$, is constant.

(a) Verify that the objective function is concave and the constraints define a concave set.

The problem shoud have made it explicit that the usual condition on a CES aggregator, $\theta \leq 1$, is imposed. Otherwise we don't have concavity. With $\theta \leq 1$ (including possibly $\theta < 0$), you might have just pointed out that CES aggregators are concave and that the objective function is therefore a linear combination, with positive weights, of concave functions, which must therefore itself be concave. The second derivative of $(x^\theta + y^\theta)^{1/\theta}$ with respect to x is easily verified to be negative, and since the function displays constant returns to scale, its second derivative matrix will have zero determinant. So it is concave.

(b) Derive the Euler equations and the transversality condition for the problem.

Assuming $W_t > 0$ not binding, the Euler equations are:

$$\partial C_t: \quad \left(1 + \left(\frac{C_{t-1}}{C_t}\right)^\theta\right)^{(1-\theta)/\theta} + \beta\left(1 + \left(\frac{C_{t+1}}{C_t}\right)^\theta\right)^{(1-\theta)/\theta} = \beta R \lambda_{t+1}$$

$$\partial W_t: \quad \lambda_t = \beta R \lambda_{t+1}$$

The TVC, applying (11) in the notes and simplifying notation with $x_t = C_t/C_{t-1}$, is

$$\limsup_{t\to\infty} \beta^t E\left[\left(1 + x_t^{-\theta}\right)^{(1-\theta)/\theta}(\hat{C}_t - C_t) - \lambda_t(\hat{W}_t - W_t)\right] \leq 0\,.$$

But the FOC's can be solved for λ_t, which allows us to rewrite the TVC as

$$\limsup_{t\to\infty} \beta^t E\Big[\left(\left(1 + x_t^{-\theta}\right)^{(1-\theta)/\theta}\right)(\hat{C}_t - C_t - \hat{W}_t + W_t) - \beta(1 + x_{t+1}^\theta)^{(1-\theta)/\theta}(\hat{W}_t - W_t)\Big] \leq 0\,.$$

Here the "hats" denote feasible alternatives to the candidate for an optimum.

(c) Show that the conditions for applying the "standard" transversality condition are not met.

There is a lagged variable in the objective function, which violates one of the conditions.

(d) Verify that the Euler equations are satisfied on any path on which the growth rate of consumption is constant.

This isn't true unless $R\beta = 1$, which should have been made explicit. But if this condition holds, the left-hand side of the ∂C Euler equation is constant, so λ is constant, and both Euler equations hold.

(e) Determine a set of (W_{-1}, C_{-1}) pairs for which a constant-growth path is feasible and satisfies the transversality condition.

The transversality condition, with constant C growth, is β^t times a constant times $\hat{C}_t - \hat{W}_t - C_t + W_t$, plus a constant times β^t times $W_t - \hat{W}_t$. Looking first at the first of these two terms, since $W_t > 0$ for all t, $C_t < W_t + \bar{Y}/R$ for all t and therefore $\hat{C} - \hat{W}$ is bounded above for all t. So this first term goes to zero so long as $W_t - C_t$ does not blow up at β^{-t} or greater rate. Looking at the second term, because feasible W_t's are positive, that term goes to zero unless W_t itself grows at rate β^{-t} or faster. But whenever $C_t = (1 - R^{-1})W_t + \bar{Y}/R$, we have $W_{t+1} = W_t$. So if $R\beta = 1$, any $\bar{C}, \bar{W}$ pair satisfying $\bar{C} = (1-\beta)\bar{W} + \beta\bar{Y}$ Will be feasible and satisfy the Euler equations and the TVC. In fact, any constant value of C_t/C_{t-1} less than R will satisfy the Euler equations, and by choosing the initial C_0 in the correct proportion to W_0, we can sustain such a growth rate without violating transversality. For a given W_0, non-explosiveness of a fixed-x path implies

$$C_0 = (1 - \bar{x}\beta)W_0 + \beta\frac{1 - \bar{x}\beta}{1 - \beta}\bar{Y} \, .$$

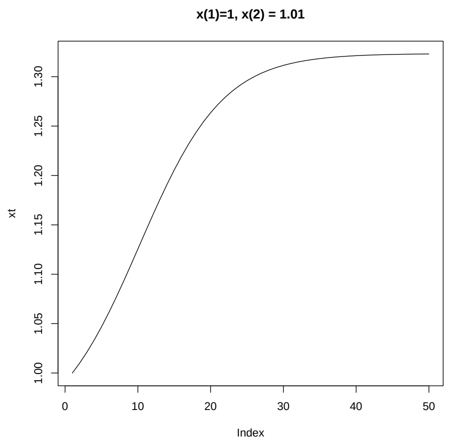
(See Jae Won's slides for details of the argument for this.) Obviously in this equation, lower initial C's correspond to higher steady growth. Note, though, that at the initial date we have given values of both W_0 and C_{-1}. For given W_0, C_{-1}, lower values of C_0 imply lower values of x_0, while for constant-growth paths lower values of C_0 imply higher $\bar{x}$ values (and hence x_0 values). There is therefore at most one constant growth path consistent with any single initial condition.

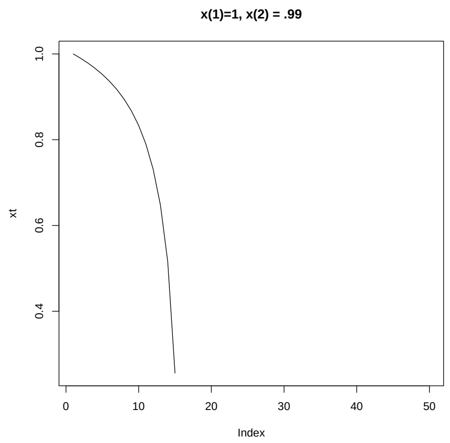
(f) Derive a difference equation in $x_t = C_t/C_{t-1}$ that is satisfied on any path that satisfies the Euler equations.

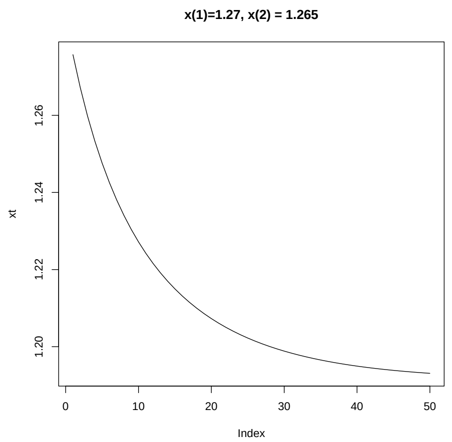
$$(1 + x_t^{-\theta})^{(1-\theta)/\theta} + \beta(1 + x_{t+1}^{\theta})^{(1-\theta)/\theta} = (1 + x_{t+1}^{-\theta})^{(1-\theta)/\theta} + \beta(1 + x_{t+2}^{\theta})^{(1-\theta)/\theta}$$

Of course for the simulations below it is easiest if this is solved for x_{t+2} as a function of x_t and x_{t+1}, which is straightforward to do.

(g) Setting $\theta = -1$, $\beta = .9$, and $R = 1/\beta$, plot paths, over 30 or so periods (long enough to show any limiting behavior that might be present) starting from various non-constant initial conditions. Consider some with $x_0 < x_1$, some with $x_0 > x_1$; and with various degrees of closeness of x_0/x_1 to 1.







(h) State whether you think there is a unique optimum path for every set of initial conditions and explain your reasoning. If you can prove analytically that there is or is not a unique optimum, you get extra credit.

It's clear that some initial conditions (very high C_{-1}/W_0, for example) are inconsistent with any constant growth solution. Whether there is always some feasible solution, I'm not sure. The simulations plotted above show convergence to a constant growth rate in some cases. This suggests that some initial conditions could be consistent both with constant-growth solutions and with non-constant growth solutions that satisfy transversality, implying non-uniqueness.