ANSWERS FOR RISK SHARING EXERCISE

In class we discussed a simple model with two countries (or two types of agents) in which each country i's representative agent solves

$$\max C_{it}, B_{it} \sum_{t=0}^{\infty} \beta^t U(C_{it}) \quad \text{subject to} \tag{1}$$

$$C_{it} + B_{it} = R_{t-1} B_{i,t-1} + Y_{it} \tag{2}$$

and the traded risk-free asset B is in zero net supply. The lecture characterized the solution of the linearized model for this case, without giving a full derivation. In the linearized model, with Y_t i.i.d. with support included in $\mathbb{R}^+$, C and B are both martingales, implying they do not converge and therefore must eventually violate any boundedness conditions on them that are present in the original nonlinear version of the model.

(i) Assume $U() = \log()$. Solve the model linearized around $B = 0$ and verify the conclusions in the lecture notes. A solution will specify C_{it} and B_{it} as functions of lagged data and current exogenous shocks (here Y_{it} itself).

FOC's are

$$\partial C : \qquad \frac{1}{C_{it}} = \lambda_{it}$$

$$\partial B : \qquad \lambda_{it} = \beta R_t E_t \lambda_{i,t+1}$$

The market-clearing condition is $B_{1t} + B_{2t} = 0$. The TVC is $\limsup \beta^t E[B_{it}/C_{it}] \leq 0$, and since $B_{1t} = -B_{2t}$, this means that $E[\beta^t B_{it}/C_{it}] \to 0$. Though you weren't asked explicitly to do this, it might be worth noting that trade in B cannot implement the complete-markets solution. The complete-markets solution would have C_{1t}/C_{2t} constant. Since $C_{1t} + C_{2t} = Y_{1t} + Y_{2t} = \bar{Y}_t$, this solution would require the existence of a fixed number $\theta \in (0, 1)$ such that $C_{1t} \equiv \theta \bar{Y}_t$. Then taking the difference of the two budget constraints, we get

$$2B_{1t} + (2\theta - 1)\bar{Y}_t = 2R_{t-1} B_{1,t-1} + \Delta Y_t \,,$$

where $\Delta Y_t = Y_{1t} - Y_{2t}$. Dividing the whole equation by $2C_{1t}$ gives us

$$\frac{B_{1t}}{C_{1t}} + 1 - \frac{1}{2\theta} = R_{t-1} \frac{C_{1,t-1}}{C_{1t}} \frac{B_{1,t-1}}{C_{,t-1}} + \frac{\Delta Y_t}{2\theta \bar{Y}_t} \,.$$

With Y_{it} i.i.d. across i as well as t, The last term in this expression is i.i.d. with zero mean, (The mean exists, assuming $Y_{it} > 0$ with probability one, because in that case the term in question is bounded by $\pm 1/2\theta$.). Taking expectations conditional on information at $t - 1$ gives us

$$E_{t-1}\left[\frac{B_{1t}}{C_{1t}}\right] + 1 - \frac{1}{2\theta} = \beta^{-1} \frac{B_{i,t-1}}{C_{i,t-1}} \,.$$

The only solution to this equation that satisfies the TVC is

$$\frac{B_{1t}}{C_{1t}} \equiv \frac{(2\theta - 1)\beta}{2\theta(1 - \beta)} \,.$$

But a constant B_{1t}/C_{1t} is incompatible with maintaining fixed shares of C_{1t} and C_{2t} in the aggregate $\bar{Y}_t$. With B the only traded asset, borrowing and lending would be needed to maintain the fixed ratios of C's, and that would violate the fixed B/C ratio. The FOC's tell us that with the ratio of C_{it} to $\bar{Y}_t$ fixed and $\bar{Y}_t$ i.i.d.,, R_t will be inversely proportional to $\bar{Y}_t$. To simplify notation, let $C_{1t} = \kappa_0 B_{1t}$ and $R_t = \kappa_1/C_{1t}$. Then the country 1 budget constraint is

$$(1 + \kappa_0)B_{1t} = \kappa_1\kappa_0 + Y_{it}$$

Dividing by C_{it} we get

$$\frac{1}{\kappa_0} + 1 = \frac{\kappa_0\kappa_1}{\theta\bar{Y}_t} + \frac{Y_{it}}{\bar{Y}_t}$$

But the right-hand-side of this equation can be constant only if Y_{2t} itself is constant. In other words, the bonds-only competitive equilibrium cannot support the complete-markets allocation unless the Yit's are constant, rather than randomly varying.

You could linearize or log-linearize (except that because the steady-state B is zero, that variable has to be left as a level. The mean of Y_{it} is $\hat{Y}$, which is also the deterministic steady state mean of c_{it}. The model linearized w.r.t. logs of all variables but B, once the Lagrange multipliers have been eliminated from the FOC's, is

$$dc_{it} = E_t[dc_{i,t+1}] - dr_t$$
$$\hat{Y}\,dc_{it} + dB_{it} = \beta^{-1}dB_{i,t-1} + \hat{Y}\,dy_{it}$$
$$dB_{1t} + dB_{2t} = 0\,.$$

This gives us (as $i = 1, 2$) 5 equations in the 5 variables dc_1, dc_2, dB_1, dB_2, and dr. This system can be solved analytically fairly easily. Adding the two budget constraints we see that $dc_{1t} + dc_{2t} = dy_{1t} + dy_{2t}$. Then adding the two FOC's and using the facts that deviations from steady state have zero expectation and that the dy's are i.i.d., we can see that

$$dr_t = -\frac{1}{2}(dy_{1t} + dy_{2t})\,.$$

Plugging this back into the country 1 FOC, we get

$$E_t dc_{1,t+1} = dc_{1t} - \frac{1}{2}(dy_{1t} + dy_{2t})\,.$$

Using this to solve forward the individual budget constraint, we arrive at

$$\hat{Y}\,dc_t = (\beta^{-1} - 1)dB_t + \hat{Y}\frac{1}{2}(dy_{1t} + dy_{2t})\,,$$

which is reminiscent of the standard LQ permanent income solution. Using this result and taking the difference of the two budget constraints, we can arrive at

$$dB_{1t} = dB_{1,t-1} + \beta\hat{Y}\frac{1}{2}(dy_{1t} - dy_{2t})\,.$$

This last result is the one that shows that dB in this linearized solution is a martingale with stationary increments, and hence does not remain bounded.

We noted in the lecture that in the original nonlinear model, neither agent can in fact issue truly risk-free debt, so the linearized model has to be thought of as approximating a model in which there is default risk, but only at levels of B much higher than the initial $B = 0$ level about which we linearize.

(ii) Consider the same model, but now with the exogenous Y_{it} process following $Y_{it} = Y_{i,t-1}^{\theta}\varepsilon_{it}$, where ε_{it} is i.i.d. with $E_{t-1}\varepsilon_{it} = 1$. Solve the linearized model and show that now the linearized model shows no borrowing or lending in equilibrium. Why does this change in the exogenous process for Y make such a difference?

The analytic solution follows that above, concluding that

$$\hat{Y}\,dc_{1t} = (\beta^{-1} - 1)dB_t + \hat{Y}(1-\theta)\tfrac{1}{2}(dy_{1t} + dy_{2t}) + \hat{Y}\frac{1-\beta}{1-\beta\theta}dy_{1t}$$

$$dB_{it} + \beta\hat{Y}\left(\left(\frac{1-\theta}{2} + \frac{1-\beta}{1-\theta\beta}\right)dy_{1t} + \frac{1-\theta}{2}dy_{2t}\right) = dB_{i,t-1} + \beta\hat{Y}dy_{1t}\,.$$

This matches the solution in the previous part when $\theta = 0$, but when $\theta = 1$, all the terms in y drop out, leaving $dB_{it} = dB_{i,t-1}$. This implies that whatever debt we start with, remains constant. The reason is that, because shocks to income are permanent, agents adjust their consumption fully immediately. There is no pure smoothing over time to be done. With $\theta < 1$, high income today implies lower income tomorrow, so agents will want to save some of the current high income. With $\theta = 1$ there is no such motive. Notice that in a complete markets equilibrium instead C would be a constant fraction of $Y_1 + Y_2$. But the bond market equilibrium cannot implement this allocation. It can only smooth over time, not across "states". Smoothing across states of nature requires an insurance market, or traded securities whose payoffs respond to the state.

Now consider a model in which, instead of an endowment that arrives exogenously, there is a technology in each country that uses capital to produce the consumption good, but that still only the risk-free bond is traded. Specifically, the country i budget constraint is now

$$C_{it} + K_{it} + B_{it} = A_{it}K_{i,t-1}^{\alpha} + R_{t-1}B_{i,t-1}\,, \tag{3}$$

where $\alpha \in (0,1)$.

(iii) Solve the linearized model for the case where A_{it} is i.i.d. with mean 1. Continue to assume B is in zero net supply. Linearize around $B_{it} = 0$. Is there borrowing in this linearized model?

Here an analytic solution may be possible to the linearized model, but probably not worth the effort. The difficulties I and a student commentator had with the computations show that even getting all the terms right in the input to gensys is challengingly tedious. On systems any bigger than this, one should always check algebraic calculations of derivativees entering a linearization against numerical derivatives, and where possible use computer algebra to get the derivatives. Matlab has a facility for handling symbolic expressions that can do the algebra of differentiation for you (though this may require an add-on package) and R has a `deriv()` function that will differentiate algebraic expressions.

The FOC's, after substituting out the Lagrange multipliers, are

$$\partial B: \qquad \frac{1}{C_{it}} = \beta R_t E_t\left[\frac{1}{C_{i,t+1}}\right]$$

$$\partial K: \qquad \frac{1}{C_{it}} = \beta E_t\left[\frac{A_{i,t+1}\alpha K_{it}^{\alpha-1}}{C_{i,t+1}}\right]\,.$$

In deterministic steady state, $R_t = \beta^{-1} = \alpha\bar{A}\bar{K}^{\alpha-1}$. Therefore, with $\bar{A} = 1$, $\bar{K} = (\alpha\beta)^(1-\alpha)$. The budget constraint then gives us that $\bar{C} = \bar{K}^{\alpha} - \bar{K}$.

Linearizing around this steady state with respect to the logs of all variables except B, and w.r.t. the level of B, Produces the following array for g0 and g1, the first two arguments to `gensys`:

$$
\mathtt{g0} = \begin{array}{ccccccc}
B_1 & B_2 & C_1 & C_2 & K_1 & K_2 & R \\
\hline
1 & 0 & \bar{C} & 0 & \bar{K} & 0 & 0 \\
0 & 1 & 0 & \bar{C} & 0 & \bar{K} & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{C}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{C}^{-1} & 0 & 0 & 0 \\
0 & 0 & \bar{C}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{C}^{-1} & 0 & 0 & 0
\end{array}
$$

$$
\mathtt{g1} = \begin{array}{ccccccc}
B_1 & B_2 & C_1 & C_2 & K_1 & K_2 & R \\
\hline
\beta^{-1} & 0 & 0 & 0 & \alpha \bar{K}^{\alpha} & 0 & 0 \\
0 & \beta^{-1} & 0 & 0 & 0 & \alpha \bar{K}^{\alpha} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{C}^{-1} & 0 & 0 & 0 & \bar{C}^{-1} \\
0 & 0 & 0 & \bar{C}^{-1} & 0 & 0 & \bar{C}^{-1} \\
0 & 0 & \bar{C}^{-1} & 0 & (\alpha-1)/\bar{C} & 0 & 0 \\
0 & 0 & 0 & \bar{C}^{-1} & 0 & (\alpha-1)/\bar{C} & 0
\end{array}
$$

The `pi` matrix, giving the coefficients on expectational errors, has an identity matrix in its lower four rows and zero elsewhere, reflecting the fact that the last four equations are FOC's and therefore contain expectational errors, whereas the first three do not. It is true that $E_t A_{i,t+1} = 1$, so that in equilibrium the expectational errors on the 4th and 6th will be identical, as will those on the 5th and 7th. Some students thought that suggested one could use as a `pi` matrix a 7×2 matrix with two copies of a 2×2 matrix in the last four rows. The trouble with this is that if we take the difference of the 4th and 6th equations at time zero, it implies that the marginal product of capital and the interest rate were equal at time $t = -1$. `gensys` assumes that variables dated earlier than time 0 are unrestricted and do not have to satisfy equilibrium conditions. Since the system specified this way implies a restriction on initial conditions, `gensys` gives a non-existence message, as it always does when a solution does not exist except for restricted sets of initial conditions. It is a design problem with `gensys` that it seems to encourage cleverness in specifying `pi`. In practice, it is I think always best to give each expectational equation a distinct, unrestricted error. If the system implies restrictions on the expectational errors, `gensys` will impose them automatically.

The `psi` matrix of coefficients on exogenous disturbances is 7×2, with nonzero entries in the first two rows, where the A_{it}'s enter the budget constraints, and in the last two rows, where they enter the K FOC's. With the given parameter values, the solution emerges with

$$\texttt{G1} = \begin{bmatrix} 0.50 & -0.50 & 0.00 & 0.00 & 0.08 & -0.08 & 0.00 \\ -0.50 & 0.50 & -0.00 & 0.00 & -0.08 & 0.08 & -0.00 \\ 0.96 & 0.84 & -0.00 & -0.00 & 0.16 & 0.14 & -0.00 \\ 0.84 & 0.96 & -0.00 & -0.00 & 0.14 & 0.16 & -0.00 \\ 0.90 & 0.90 & -0.00 & -0.00 & 0.15 & 0.15 & -0.00 \\ 0.90 & 0.90 & -0.00 & -0.00 & 0.15 & 0.15 & -0.00 \\ -0.63 & -0.63 & 0.00 & 0.00 & -0.11 & -0.11 & 0.00 \end{bmatrix}$$

$$\texttt{impact} = \begin{bmatrix} 0.28 & -0.28 \\ -0.28 & 0.28 \\ 0.53 & 0.47 \\ 0.47 & 0.53 \\ 0.50 & 0.50 \\ 0.50 & 0.50 \\ -0.35 & -0.35 \end{bmatrix}$$

Notice that, as can be shown analytically, the two K's are equal in equilibrium: the 5th and 6h rows of both `G1` and `impact` are identical. With this in mind, we see that the first two rows of `G1` and `impact` imply that B is a martingale driven by the difference of the two productivity shocks. So there is borrowing and lending in equilibrium.

(iv) Repeat the analyis in (iii), but now assuming $A_{it} = A_{i,t-1}^{\theta}\varepsilon_{it}$, where ε_{it} is i.i.d. with $E_{t-1}\varepsilon_{it} = 0$ and $\theta \in (0, 1)$.

One could do this by adding the A_{it}'s as additional states, but one can also do it by using the forward part of the `gensys` solution. The solution does not assume any serial correlation properties for exogenous disturbances. If there is serial correlation in them, their expected future values enter the solution via `ywt`, `fmat`, and `fwt`. While doing this efficiently may require unreasonable amounts of matrix algebra cleverness for your first exposure to these methods, showing how to do this may be more instructive than showing how to expand the g0 and g1 matrices.

The expected future values of the exogenous variables enter the system as the term

$$\Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z E_t[z_{t+s}] \,,$$

in the notation of Sims (2001), where Θ_y is `ywt`, Θ_f is `fmat`, and Θ_z is `fwt`. Using Matlab or R to generate an eigenvalue decomposition of Θ_f, and using the fact that in our case $E_t z_{t+s} = \theta^s z_t$, we can reduce the expression for the impact of future values to

$$\sum_{s=1}^{\infty} \begin{bmatrix} .28 & 0 \\ -.28 & 0 \\ -.68 & .45 \\ .68 & .45 \\ 0 & -1.13 \\ 0 & -1.13 \\ 0 & .80 \end{bmatrix} \begin{bmatrix} .95^s.9^s & 0 \\ 0 & .285^s.9^s \end{bmatrix} \begin{bmatrix} -.049 & .049 \\ .63 & -.63 \end{bmatrix} \begin{bmatrix} da_{1t} \\ da_{2t} \end{bmatrix} ,$$

where the da_{it}'s are log deviations from steady state for the A_{it}'s. Once we take the geometric sums in this expression, it will just alter the coefficients in the `impact` matrix. The `G1` matrix is unaffected. That only contemporaneous effects of exogenous variables enter the system here reflects the assumption that the exogenous variables are first-order autoregressive. With higher-order serial dependence in them, additional lags of those variables would enter.

(v) In the setup of (iii) and (iv), is there the same problem as in the classroom problem that individuals cannot in fact issue truly risk-free debt securities? Why or why not.

In the linearized models, you can if you like assume numerical values for the parameters and use `gensys` to find the solution. Use $\alpha = .3$, $\beta = .95$, $\theta = .9$, and a non-stochastic steady-state Y_{it} equal to 1.

References

Sims, C. A. (2001): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1–20, http://www.princeton.edu/~sims/.