RANDOM LAGRANGE MULTIPLIER AND TRANSVERSALITY EXERCISE

(1) The standard analytically solvable stochastic growth model is

$$\max_{\{K_t, C_t\}} E\left[\sum_{t=0}^{\infty} \beta^t \log C_t\right] \qquad \text{subject to}$$

$$C_t + K_t = A_t K_{t-1}^{\alpha}\,, \quad t = 0, \ldots, \infty$$

$$K_{-1} \text{ and the distribution of } \{A_t\} \text{ given.}$$

(a) Find the Euler equation first order conditions for a solution.
These are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t$$

$$\partial K: \qquad \lambda_t = \beta E_t[\lambda_{t+1}] \alpha K_t^{\alpha-1}\,.$$

(b) Verify that there is a solution to the Euler equations and the constraint that makes C_t proportional to K_t at every t.
Guess the solution as $C_t = \gamma K_t$. Then the FOC's and constraint, with λ solved out, can be rewritten

$$\frac{1}{\gamma K_t} = \beta \alpha E_t\left[\frac{A_{t+1} K_t^{\alpha-1}}{\gamma K_{t+1}}\right] \tag{1}$$

$$(1+\gamma) K_t = A_t K_{t-1}^{\alpha}\,. \tag{2}$$

Using (2) in (1), we get

$$\frac{1}{\gamma K_t} = \beta \alpha \frac{1+\gamma}{\gamma K_t}$$

and thus conclude that $\gamma = (1 - \beta\alpha)/(\beta\alpha)$.

(c) Verify that the conditions for the application of the standard TVC are present and that it is satisfied at the solution you have found with C proportional to K. Do you have to invoke regularity conditions on the stochastic process followed by A_t in order to get your result? If so, what are they?
The only variable that enters the problem with a lag is K_t, so it plays the role of S_t. The constraint can be solved, simply by moving C_t over to the right-hand-side, as required for applying the standard TVC. It is certainly feasible to let K_t go to zero, even though doing so will force $C_t \to 0$ also, so is likely to be a bad idea. The problem does not say explicitly that K_t has to stay positive, but this is meant to be implicit in the production function, which, for $0 < \alpha < 1$, produces imaginary numbers if we try to evaluate it at negative

K. The constraint is naturally interpreted as an inequality of the required ($\leq$) sense, even though the problem statement (mistakenly) said it was an equality. One could also observe that the $\leq$ side of the equality constraint will never be binding, so the problem is unchanged if we replace the equality constraint with an inequality. Finally K does not appear in the objective function. So the conditions for applying the standard TVC are met. It is, in this case, $E[\beta^t K_t/C_t] \to 0$. Since on our solution path K_t/C_t is constant, the TVC is trivially satisfied. To be sure that this guarantees we have a solution, we have to check that the objective function is concave and that the constraint set is convex. The objective function is concave because the $\log$ function is concave. So long as the objective function is defined for the two random sequences C_{1t} and C_{2t}, it is clear then that $\theta V(C_1) + (1-\theta)V(C_2) \leq V(\theta C_1 + (1-\theta)C_2)$, where V is the objective function and θ is a number in $(0,1)$, which is the definition of concavity. Each constraint is of the form $g(K_t, C_t, K_{t-1}, A_t) \leq 0$, with g a convex function in its first three arguments. This implies that the set of K_t, C_t, K_{t-1} values it defines is convex. Since the intersection of (even a countable infinity of) convex sets is convex, the full constraint set is convex.

(d) (*Extra credit in case you are aching to apply your rusty real analysis tools.*) Define a linear space whose points map one to one into pairs of $\{C_t\}$ and $\{K_t\}$ sequences and a metric on the space such that the objective function in this problem is continuous and the constraint set is contained entirely in the linear space. Verify the concavity of the objective function and the convexity of the constraint set. Find the continuous linear functional that separates the constraint set from the set of points preferred to the optimum. Discuss how this functional is related to the Euler equations and the TVC.

Pick the space to be that of the sequences $\{\log C_t\}, \{K_t\}$ with the norm

$$\|\{X_t\}, \{Y_t\}\| = E\left[\sum \beta^t |\log C_t| + \sum \beta^t |K_t|\right].$$

On this space, the objective function is linear and thus obviously concave. Since K_t does not enter the objective function, the tangent plane to the objective function at the candidate optimum $\bar{C}_t$ is

$$dV(dC, dK) = E\left[\sum_t (\beta^t d\log C_t + 0 \cdot dK_t)\right], \tag{3}$$

assuming this is actually a continuous linear functional under our chosen norm. But since the absolute value of this functional is clearly less than $\|\{C_t\}, \{X_t\}\|$, the function is clearly continuous at zero, which is of course all that we need to show to demonstrate continuity of a linear function. The only remaining problem is to show that at the candidate optimum $\{\bar{C}_t\}, \{\bar{K}_t\}$, every point in the constraint set satifies $dV(dC, dK) \leq 0$. The easiest way to proceed here is to use the results in the extended random Lagrange multiplier notes. Let

$$dg_t(d\log C_t, dK_t, dK_{t-1}, dA_t) = \bar{C}_t d\log C_t + dK_t - dA_t\bar{K}_{t-1}^{\alpha} - \alpha A_t \bar{K}_{t-1} dK_{t-1}$$

be the differential of the time-t constraint evaluated at the candidate optimum. Every one of these differentials is non-positive for any feasible pair of C and K sequences because of the convexity of the constraint function. So if we can construct the tangent to the preferred set as the limit of linear combinations of the dg_t's with all non-negative weights, we will have shown that the tangency function dV is non-positive for any feasible deviation from the candidate optimum. But this is just saying we want to find Lagrange multipliers, and the TVC is just the condition required to make the weighted linear combinations of finitely many dg_t functions converge to the tangency function.