CAPITAL TAX EXERCISE ANSWERS

We consider a simple frictionless growth model with capital but no labor. The government, for reasons we leave unspecified, has to finance a given stochastic time path of transfer payments g_t by use of the only tax available to it, a proportional tax on capital set aside this period for use next period.

The representative private consumer-producer solves

$$\max_{c_t,b_t,k_t} E\left[\sum_{t=0}^{\infty}\beta^t \log c_t\right] \qquad \text{subject to} \tag{1}$$

$$c_t + (1+\tau_t)k_t + b_t = a_t k_{t-1}^{\alpha} + \delta k_{t-1} + r_{t-1}b_{t-1} + g_t\,. \tag{2}$$

The government is benevolent, so it also maximizes (1), but it chooses, in addition to the private sector's choice variables, the tax rate τ_t, and the (real) interest rate r_t. It does not choose g_t — it takes that as given. The government's constraints are all the private sector constraints and FOC's, plus the government budget constraint

$$b_t + \tau_t k_t = g_t + r_{t-1}b_{t-1}\,. \tag{3}$$

We assume the exogenous shocks a_t and g_t are i.i.d., with means 1 and 0, respectively.

(a) Find the Euler equation first order conditions (FOC's)for the consumer.

$$\begin{aligned}
\partial c: &\qquad \frac{1}{c_t} = \lambda_t \\
\partial b: &\qquad \lambda_t = \beta r_t E_t[\lambda_{t+1}] \\
\partial k: &\qquad (1+\tau_t)\lambda_t = \beta E_t\left[\lambda_{t+1}(\alpha a_{t+1}k_t^{\alpha-1} + \delta)\right].
\end{aligned}$$

We can substitute out the λ's to obtain

$$\begin{aligned}
\mu: &\qquad \frac{1}{c_t} = \beta r_t E_t\left[\frac{1}{c_{t+1}}\right] \\
\nu: &\qquad \frac{1+\tau_t}{c_t} = \beta E_t\left[\frac{\alpha a_{t+1}k_t^{\alpha-1}+\delta}{c_{t+1}}\right],
\end{aligned} \tag{$*$}$$

where the Greek letters at the left are the Lagrange multipliers we will assign to these equations as constraints in the government's optimization problem.

(b) Find the Euler equation FOC's for the government.

Here we treat the government's constraints as the social resource constraint

$$\psi: \qquad c_t + k_t = a_t k_{t-1}^{\alpha} + \delta k_{t-1}$$

together with the government budget constraint (3) and the two private FOC's. We assign the Lagrange multiplier ω to the government budget constraint. The FOC's *then are*

$$
\begin{aligned}
\partial c: &\qquad \frac{1}{c_t} = \frac{-\mu_t + \mu_{t-1} r_{t-1}}{c_t^2} + \frac{-(1+\tau_t)\nu_t + (a_t \alpha k_{t-1}^{\alpha-1} + \delta)\nu_{t-1}}{c_t^2} + \psi_t \\
\partial b: &\qquad \omega_t = \beta r_t E_t \omega_{t+1} \\
\partial k: &\qquad \psi_t + \tau_t \omega_t = \beta E_t \left[\nu_t \alpha(\alpha - 1) \frac{a_{t+1} k_t^{\alpha-2}}{c_{t+1}} + \psi_{t+1}(\alpha a_{t+1} k_t^{\alpha-1} + \delta) \right] \\
\partial \tau: &\qquad \frac{\nu_t}{c_t} + k_t \omega_t = 0 \\
\partial r: &\qquad E_t \frac{\mu_t}{c_{t+1}} + E_t[\omega_{t+1}] b_t = 0
\end{aligned}
$$

(c) Verify that the problem has a deterministic steady state at $\tau = b = 0$ and solve for the steady state values of c, k, and r for this case. [Hint: Find formulas for steady state r and k as a function of the parameters, then express steady state c as a function of steady state k. Lagrange multipliers on private FOC's should emerge as zero in this steady state.]

That the problem has such a steady state is easily checked. If $\nu = \mu = \omega = 0$, then $\psi = 1/c_t$ from the ∂c equation. Then the ∂k equation must be satisfied, because it reduces to the same equation as the consumer's ∂k FOC. And the government budget constraint is satisfied automatically in steady state (where $g_t \equiv Eg_t \equiv 0$) when $b = \tau = 0$. Then all that remains is to verify that there is a steady-state set of values for c and k that satisfies the private agent's FOC and the social resource constraint. It is straightforward to check, using the ∂k equation that

$$\bar{k} = \left(\frac{\alpha\beta}{1 - \delta\beta} \right)^{1/(1-\alpha)} .$$

and then by substituting this value into the social resource constraint, $\bar{c} = \bar{k}^{\alpha} + (\delta - 1)\bar{k}$. Since this expression subtracts a linear term from a concave one whose slope goes to zero, it could in principle fail to deliver a positive $\bar{c}$, but at the optimal $\bar{k}$ this does not happen, as can be checked here. For the parameter values specified, $\bar{k} = 3.59$, $\bar{c} = 1.22$.

(d) Show that the system of FOC's also has a steady state at every other constant positive value of τ. In this case, it is not necessary to solve for the steady state explicitly. Just show that the equations of the system can be solved for this case. Note that for this to be true, at least one of the equations defining steady state must be redundant.

You are asked to show there are multiple solutions for steady state, and since there are as many equations as unknowns, one of the equations has to be redundant. The ∂b equation, once we have used the steady state version of $(*)$ to set $\bar{r}\beta = 1$, becomes an identity in steady state.

Suppose we set a value for τ greater than 0. Then the ν equation will determine steady-state capital, we can use the SRC as before to find steady-state c, the government budget constraint will determine steady state b, and the μ equation determines

r. There are then five equations to determine the four Lagrange multipliers, but as we have already checked, one of them is an identity, so we have four equations in four unknowns. This was as far as I really expected you to get. However, there remains the possibility that these four equations contain yet another singularity, so that there could be no solution. I've verified (though I only programmed it once, so I could be wrong) that for a tax rate of .05 and the other parameters as given in the problem, that there is a well-defined solution for all the Lagrange multipliers.

Note that the private FOC constraints are equalities, so the usual presumption that constraints, arranged so they are interpretable as inequalities, will have positive Lagrange multipliers, does not apply here. Also, the government budget constraint has debt and taxes on the left, undesirable things, so that it is interpretable as a $\geq$ inequality. It is natural then that its Lagrange multiplier ω emerges as negative.

Also, we are checking here only for solutions to the FOC's. The problem has a concave objective function, but the constraints are not everywhere convex. Thus it is possible that some or all of the steady states we have located are not optima.

(e) Assume $\alpha = .3$, $\beta = .95$, $\delta = .93$. Linearize the system around the steady state you found in (c) and use `gensys` or another linear rational expectations solver to find the impact effects of the two exogenous shocks (`impact` in the `gensys` output) and $G_1^s x$, $s = 1, \ldots, 3$, where x is the impact matrix from `gensys` and G_1 is `G1` from gensys. Together, what you have calculated will be the first four periods' responses of all variables to the exogenous shocks. Observe whether the optimal time path of the capital tax is in fact decreasing in this linearized solution.

In the neighborhood of the zero-tax steady state you were asked to study, the optimal response absorbs much of the g shock in debt issue, and allows the higher debt, and a lower capital stock, to persist. In the neighborhood of the $\tau = .05$ steady state, a similar fraction of the initial impact is absorbed in debt, but the debt is quickly brought back down to the original (non-zero) level. The capital stock declines initially, but then gradually nearly recovers to its initial (depressed, because of $\bar{\tau} > 0$) level.

(f) The result that the optimizing government might leave τ constant at a non-zero level seems to clash with Chamley's classic result. Can you explain which aspects of this model are central to the differing conclusion?

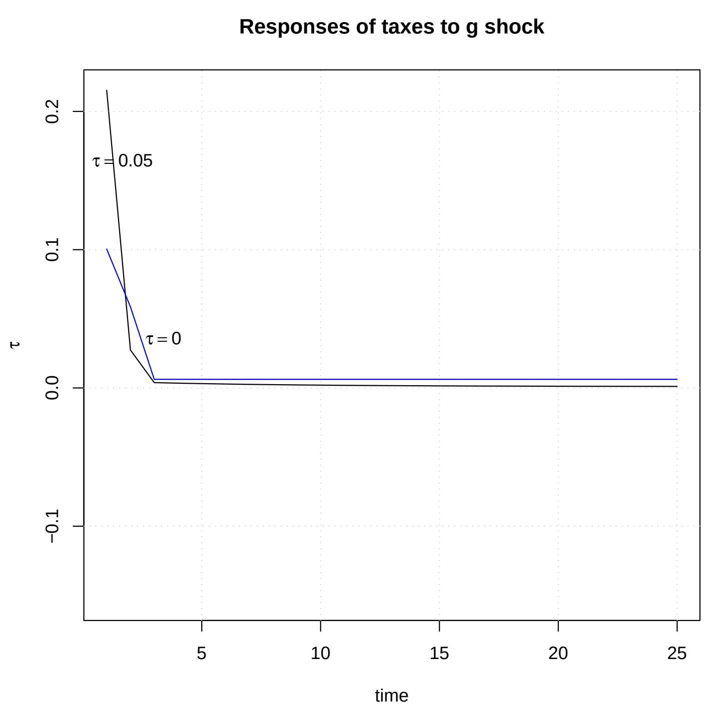
The most important difference is that in this model the capital tax is on currently produced capital, which can react immediately. Surprise capital taxes therefore are distorting, unlike the case where they can be imposed on existing capital. When taxes are already high, and debt is correspondingly high, the initial response of the tax rate has to be larger, because of the lower tax base, even though about the same fraction of the initial g shock is absorbed in debt. But the cost of the higher capital taxes is higher at the high-debt steady state, so it is optimal to make the change in taxes and debt less persistent.

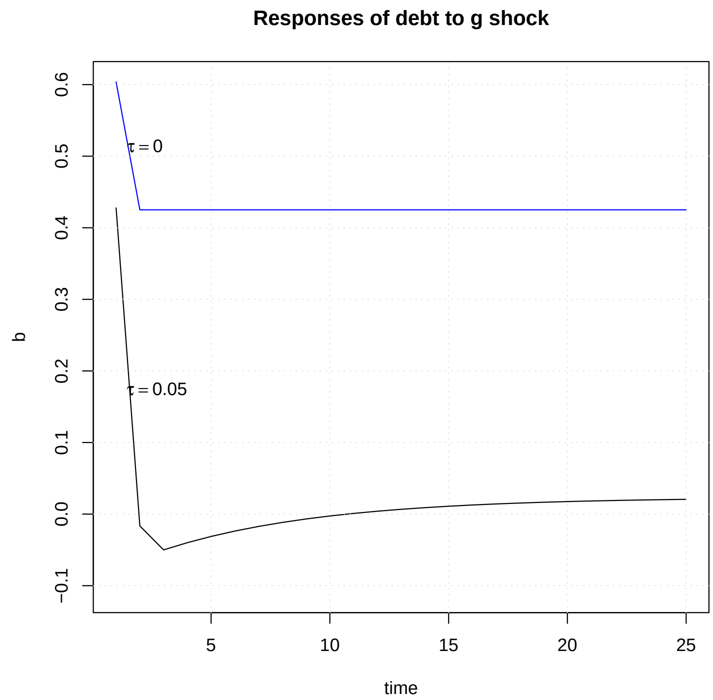
At neither steady state is it optimal (to first order) to bring debt and taxes back to the zero level. g shocks therefore have persistent effects on all variables in the economy.

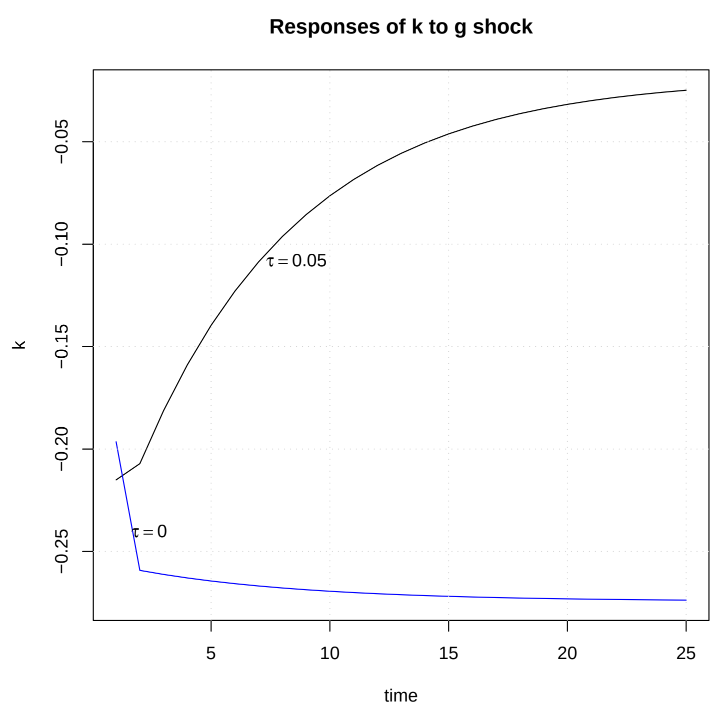
Plots of the responses are below.

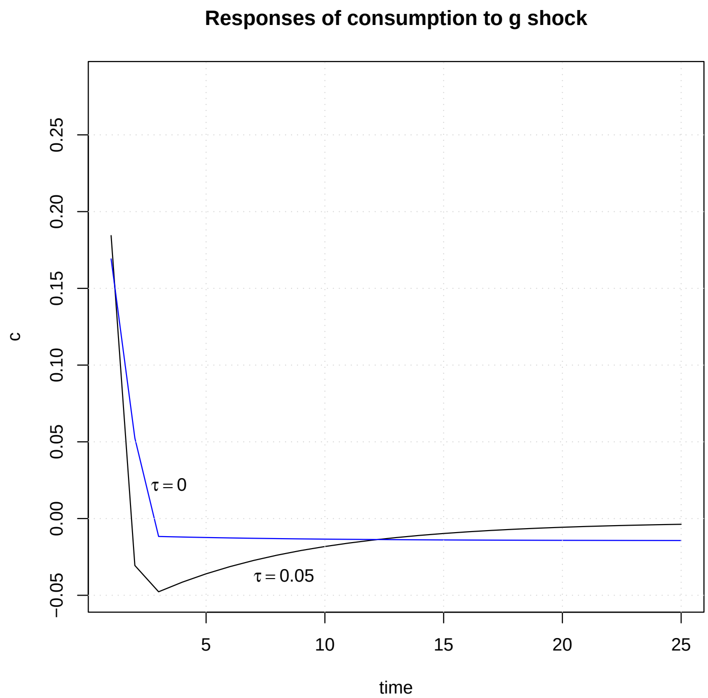
Note: I am only 85% sure that the impulse responses below are correct. I got different results when I did the problem last week, but I can't see anything wrong with the computations that produced the results below. There was one additional error in `ktaxsys`. The sign of the Lagrange multipliers `mu` and `mul` in the `dc` expression were incorrect. Lagrange multiplier signs are of course arbitrary, but they have to be treated consistently across equations. This repair seemed to have only minor effects on the solution at the zero-tax steady state.

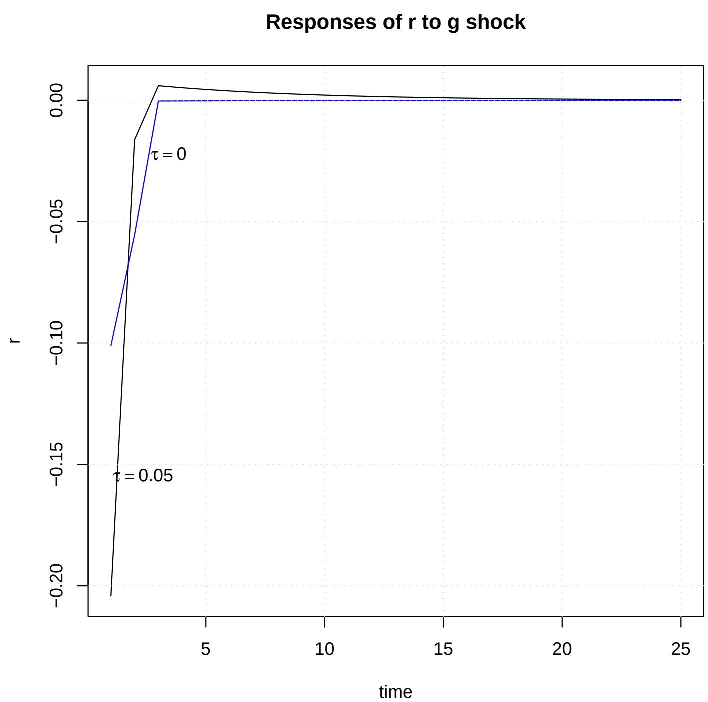
It might seem paradoxical that debt issued in the first period is brought back down again so promptly with so little apparent tax effort after the first period. The explanation is that real interest rates are pushed negative in the first period. That is, the debt, instead of increasing if there are no surpluses to shrink it, shrinks by itself because of the negative net real interest rate. Implementing such a policy in a world with a price level and a zero lower bound on nominal interst rates would require substantial anticipated inflation.




Responses of debt to g shock
b
0.6
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
τ = 0
τ = 0.05
5
10
15
20
25
time


Responses of k to g shock
k
−0.05
−0.10
−0.15
−0.20
−0.25
τ = 0.05
τ = 0
5
10
15
20
25
time


Responses of consumption to g shock
c
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
τ = 0
τ = 0.05
5
10
15
20
25
time


Responses of r to g shock
r
0.00
−0.05
−0.10
−0.15
−0.20
τ = 0
τ = 0.05
5
10
15
20
25
time