FINAL EXAM

Answer all questions. The exam has 120 points, with each point meant to correspond to one minute. This makes the weight of this exam match that of the midterm/first-half-final. However, we have the room for three hours, and you can work on the exam for up to three hours. Still, be careful as usual not to devote disproportionate time to any one question until you have attempted them all.

(1) (45 minutes) Consider an economy in which the representative agent maximizes

$$E\sum_{t=0}^{\infty}\beta^t \log(C_t) \qquad \text{subject to} \tag{1}$$

$$C_t(1+\gamma v_t)+\tau_t C_t+\frac{M_t}{P_t}=Y_t+\frac{M_{t-1}}{P_t}\,. \tag{2}$$

The choice variables are consumption C and money M. Velocity v is defined as $v_t = P_t C_t/M_t$. τ is the consumption tax rate and Y_t is an i.i.d. endowment process with $Y_t > 0$ with probability 1. β is of course between zero and one.

The government budget constraint is

$$\frac{M_t}{P_t}+\tau_t C_t=\frac{M_{t-1}}{P_t}\,. \tag{3}$$

(a) Suppose policy sets the growth rate of money M_t/M_{t-1} equal to a constant G, which may be greater or less than one. Show that in this case there are values of G for which there is a uniquely determined price level. Characterize the set of G's for which there is an equilibrium with a uniquely determined price level. [Note: Though uniquely determined, the price level will change over time in at least some of these equilibria.]

The FOC's of the private agent are

$$\frac{1}{C_t}=\lambda_t(1+2\gamma v_t+\tau_t)$$

$$\frac{\lambda_t}{P_t}(1-\gamma v_t^2)=\beta E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right]\,.$$

Let

$$Z_t=\frac{M_t}{P_t C_t(1+2\gamma v_t+\tau_t)}\,.$$

From the government budget constraint (3) we can solve for τ_t to arrive at

$$\tau_t=\frac{1-G}{v_t G} \tag{‡}$$

$$\therefore Z_t=\frac{1}{v_t+2\gamma v_t^2+G^{-1}-1}\,. \tag{$*$}$$

Using these substitutions in the FOC's leads us to

$$Z_t(1-\gamma v_t^2) = G^{-1}\beta E_t Z_{t+1} \tag{$\dagger$}$$

Notice from $(*)$ that Z is monotone decreasing in v. As $v \to \infty$, $Z \to 0$. If $G = 1$, then as $v \to 0$, $Z \to \infty$. But if $G > 1$, v approaches a finite lower bound as $Z \to \infty$, while if $G < 1$, Z reaches a finite upper bound as $v \to 0$. We can rule out $Z < 0$ in equilibrium, because this would require $\tau < -(1+2\gamma v)$. Once τ reaches the subsidy rate $-(1+2\gamma v)$, consumption becomes "free" at the margin. That is, the subsidy rate on consumption is so high that it more than offsets the direct costs of consumption, and wealth can be increased by increasing consumption. This is obviously inconsistent with equilibrium.

So long as $G^{-1}\beta < 1$, the Z difference equation ($\dagger$) behaves like the one we discussed in class, so a similar argument shows that there is only one solution in which Z remains bounded away from zero and infinity with probability one. Z getting arbitrarily close to zero requires v getting arbitrarily large, and as in our classroom model this would make the $(1-\gamma v^2)$ term on the left of ($\dagger$) negative, which is inconsistent with equilibrium (as the rhs must be positive). Z getting arbitrarily large with $\beta < G < 1$ implies v getting arbitrarily close to zero. This implies that real wealth in the form of money balances grows arbitrarily large relative to consumption. This implies τ gets arbitrarily large, from ($\ddagger$). It was acceptable at this point to postulate that $\tau > 1$ is impossible, as it is true that it is hard to imagine a consumption tax rate that exceeds 100%. However, in this model $\tau > 1$ is not really impossible. It occurs on paths in which M/P is also growing arbitrarily large and $C = Y$ is being maintained. Agents would be able to pay the high taxes out of their high wealth, and they would perceive the high wealth as needed to meet the ever-growing burden of future taxes. The standard transversality condition does not apply here, because it assumes that it is always feasible to reduce wealth to zero and proceed on some feasible path. Here, on paths where τ is perceived as exploding, it will not appear feasible to agents to reduce M/P to zero, or even at all. The conclusion is that with $\beta < G < 1$, paths where v gets arbitrarily small are ruled out if τ is bounded above, because they require unbounded tax rates. If τ is not bounded above, the policy is feasible but does not lead to a unique price level, because a range of initial prices below that consistent with constant v are all consistent with deflationary equilibria and exploding tax rates.

If $G > 1$, $Z \to \infty$ requires $v + 2\gamma v^2 + G^{-1} - 1 \to 0$, from $(*)$. That is, the marginal cost of consumption goes to zero. Here, since it is subsidies, not taxes, that are increasing as v decreases, the usual sort of transversality argument applies. Individuals on paths where v is decreasing see the reduction in next-period wealth from consuming a unit of their growing (though bounded) current wealth getting ever smaller, approaching zero. This cannot be an equilibrium. Eventually, the bounded discounted present value of transaction cost benefits of holding holding a unit of M/P into next period must be dominated by the (subsidy-enhanced) current utility obtainable from turning it into current consumption.

If $G < \beta$, the difference equation in Z is stable, so any initial value of Z is consistent with bounded Z. However, the difference equation implies $E_t[Z_{t+s}] \to 0$ as $s \to \infty$. This implies that Z must get arbitrarily small with positive probability, and as we have already observed this implies $1 - \gamma v^2 < 0$ and is therefore inconsistent with equilibrium. So there is no equilibrium for $G < \beta$.

Depending on whether you assume the consumption tax rate is bounded above or not, the answer is then that there is a unique price level when $G > \beta$, or when $G > 1$. There is no upper bound on the G's consistent with a uniquely determined price level.

(b) If the policy authority sets $G > \beta^{-1}$, is there an equilibrium? Can welfare be improved by changing G? If instead the authority sets $G < \beta$, is there an equilibrium? Can welfare be improved by changing this G? Is there an optimal value of G? [Hint: Consider the relation of v to C in equilibrium, by looking at the social resource constraint.]

$G > \beta^{-1}$ raises no problems for existence of equilibrium. Welfare in this model is determined by v. In the unique equilibria with $G > \beta^{-1}$, equilibrium v is determined by the equation $(1-\gamma v^2) = G^{-1}\beta$. The smaller is G, therefore, the smaller is v, and thus by the SRC $C(1 + \gamma v) = Y$, the larger is C and welfare. So for this range of G values is is always possible to improve welfare by reducing G. With $G < \beta$, as we verified above, there is no equilibrium. In other words, discussing whether welfare improvement is possible is irrelevant, since there is no economy in which $G < \beta$ forever.

(c) If the policy authority instead sets the tax rate τ at some fixed value, are there values of τ for which there is a uniquely determined price level?

Here the same Z_t, expressed in terms of $\bar{\tau}$ and v, becomes

$$Z_t = \frac{1}{v_t(1 + 2\gamma v_t + \bar{\tau})}$$

and the difference equation becomes

$$(1 - \gamma v_t^2)Z_t = \beta E_t[(1 + \bar{\tau} v_{t+1})Z_{t+1}] \,.$$

This does not quite work, so we define $Z_t^* = (1 + \bar{\tau} v_t)Z_t$, which lets us write

$$\frac{1 - \gamma v_t^2}{1 + \bar{\tau} v_t} Z_t^* = \beta E_t[Z_{t+1}^*] \,.$$

It can be checked that Z^* is monotone decreasing in v, here with Z going to zero iff v goes to infinity and vice versa. The coefficent on Z^* on the left is monotone decreasing in v as in the other model, so the same sort of analysis applies: Z^* has a unique stable solution if and only if $\beta < 1$, which we assume true as a standard assumption. Again Z growing arbitrarily small and thus v arbitrarily large, is inconsistent with equilibrium because it eventually makes the left-hand side of the difference equation negative. Z can grow arbitrarily large only by v_t approaching zero. Here, since $\bar{\tau}$ is fixed, the usual transverality reasoning will apply to rule out such paths, whether τ is positive or negative. So the conclusion is that there is a uniquely determined price level for every possible value of τ.

(d) Would your conclusions be different in this problem if instead of the tax on consumption, which affects consumers' perceived tradeoff between consuming now and saving to increase money balances, taxes were lump sum? Explain why or why not.

The τ or G terms in the denominator of Z disappear if the tax is lump-sum. This means that Z goes to $\pm\infty$ exactly as v does the opposite. This in itself does not change the analysis much. The only change in the conclusions of the analysis is for the case $\beta < G < 1$, if we eliminate such equilibria with the $\tau < 1$ condition. For the model with a consumption tax, the consumption tax *rate* must be unbounded in such equilibria, whereas with a lump-sum tax the level of the tax must be unbounded. Within this model there is no barrier to arbitrarily high tax rates or tax levels, so long as the agents carry enough wealth to pay the high taxes. The conclusion that without bounds on taxes there is non-uniqueness of equilibrium in the $\beta < G < 1$ case still hold, therefore. However one might argue that lump-sum taxes exceeding total income (but not total wealth) are less implausible than consumption taxes at a rate over 100%.

(2) (45 minutes) Consider an economy in which the representative agent maximizes

$$\sum_{t=0}^{\infty} \beta^t \log(C_t) \qquad \text{subject to} \tag{4}$$

$$C_t + K_t + B_t = (1-\tau_t)AK_{t-1} + \bar{Y} + \bar{g} + R_{t-1}B_{t-1}\,. \tag{5}$$

C is consumption, K is capital, and $\bar{Y}$ is constant endowment income. The fixed gross return on capital is $A > 1$. B is (real) government debt, R is the interest rate on the debt and $\bar{g}$ is government lump-sum transfer payments.

The government budget constraint is.

$$B_t + \tau_t AK_{t-1} = \bar{g} + R_{t-1}B_{t-1}\,. \tag{6}$$

Note that there is no randomness in the economy.

(a) Show that if the government's announcements about future policy are believed and correspond to actual government future behavior, then in this full-commitment case the optimal tax policy has $\tau = 0$ in the long run.

Because the tax at date t applies to capital already in place at t, the full-commitment solution allows, without any distortion, a tax rate so high at the initial date that the government can finance all future expenditure from its initially expropriated wealth. So the optimal tax is zero not just in the long run, but at every date but the initial date.

(b) Show that for some possible values of A and a fixed tax rate $\tau_t \equiv \bar{\tau}$, equilibrium implies that K grows steadily at an exponential rate. Does this invalidate the usual conclusion that the marginal discounted utility cost of increasing $\bar{\tau}$ (and g correspondingly) from an initial zero steady state is zero? Explain why or why not. [Hint: You should be able to answer this without actually solving to find the marginal cost.]

The problem should have stated that K grows *eventually* at an exponential rate, rather than "steadily" at such a rate. I meant to convey that growth never slows, not that its rate is constant, but some students were thrown off by the wording. The FOC's can be reduced to

$$\frac{1}{C_t} = \beta(1-\bar{\tau})A\frac{1}{C_{t+1}}$$
$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}} \,.$$

The SRC is

$$C_t + K_t = AK_{t-1} + \bar{Y} \,.$$

The first FOC lets us conclude that $C_t = C_0\big(A\beta(1-\bar{\tau})\big)^t$. Plugging this in to the SRC and solving forward gives us

$$K_0 = C_0 \frac{\beta(1-\bar{\tau})}{1-\beta(1-\bar{\tau})} - \bar{Y}\frac{1}{A-1} \,, \qquad (\Diamond)$$

assuming $A^{-t}K_t \to 0$. Note that we also need some kind of no-Ponzi condition, which was not stated in the problem. The natural one here is $K_t > 0$, all t. Equation ($\Diamond$) implies that as $\tau \to 1$ K_0 eventually turns negative, so there is an upper bound on τ, beyond which all capital will be consumed in the initial period. The TVC has a standard form here, and the problem has concave objective function and convex contraint, so the TVC is $\beta^t K_t/C_t \to 0$. Along paths consistent with the Euler equations, this is equivalent to $A^{-t}(1-\bar{\tau})^{-t}K_t \to 0$. Therefore any path for which $A^{-t}K_t$ fails to converge to zero violates the transversality condition, and the solution is indeed unique. (Of course here we are using the TVC as if it were necessary, which sometimes it isn't. A direct argument would simply show that if $\beta^t K_t/C_t$ does not go to zero, but the Euler equations are satisfied, part of the ever-growing gap between K and C can be consumed now without violating feasibility conditions, thereby producing a welfare gain.)

The usual argument depends on comparing a current benefit of increased consumption to the discounted disutility of a stream of future consumption decreases. The fact that the future consumption decreases are growing over time does not prevent them, in this model, from having finite discounted present value. In fact, because utility is $\log C$, any exponentially growing path of consumption changes produces a linear path of utility changes, which will have finite present value at any discount rate less than one.

(c) Suppose that the public extrapolates current tax behavior, i.e. that they always believe that next period's τ_{t+1} will be the same as this period's τ_t. Is this likely to change the conclusions you came to in part (2a)?

The easy answer that we can make the tax arbitrarily high in the first period without any distorting effect is certainly no longer available. Higher taxes this period will produce anticipated higher taxes in the future, and hence lower investment, so an arbitrarily high tax in the current period is certainly not desirable. However, determining whether it is optimal for τ eventually to go to zero is still a difficult question, which no one found the answer to in the exam time constraints.

(d) If the public always believes that the capital tax will remain constant at the current rate, will it turn out to be optimal for the government actually to make the capital tax constant? [Hint: This part of the question is open-ended. I'm not sure it's answerable in the exam time frame. Be particularly careful not to waste too much time on this part.]

Taxes have the same effect in all periods in this setup. There is no special character to the initial period, because current taxes are extrapolated into the future in the same way in all periods. The usual source of time inconsistency is therefore gone, and a steady state with constant tax rate might exist. Probably from given initial conditions the time path of the optimal tax rate will not be constant, but here again the proof is more than could be carried through during the exam.

(3) (30 minutes)

(a) The Lucas critique attacked the large macroeconomic models that were in wide use at the time it was published. At the time, there were also monetarist models in which output and prices were regressed on current and past values of the money stock, with the results interpreted to be usable to project the effects of changes in monetary policy. Did the Lucas critique apply also to these models? Explain why or why not.

The most direct form of the Lucas critique did not apply to the monetarist models. The large models contained explicit expectational terms which were treated as adaptive. The Lucas critique in its simplest form just pointed out that as the stochastic process of the economy changed, the expectational rules would change, and the models did not allow for that.

The monetarist models did not contain explicit expectational terms. If they were interpreted as linear approximations to a reduced form of a correctly specified rational expectations model, they were therefore not subject to the simplest form of the Lucas critique. However the monetarist models were linear, and the Lucas critique can be interpreted as implying that certain forms of nonlinearity are important, at least when evaluating policies that permanently change the inflation rate. Thus the monetarist models while not shown to be self-contradictory by the Lucas critique, were shown to have a limited range of applicability.

(b) Most of the Asian countries that suffered exchange rate crises in the 90's showed relatively little inflation after the crisis. Does this contradict theories that explain the crises as due to the potential for expansive post-attack monetary and fiscal policies?

The pattern of a sudden devaluation, followed by no persistent inflation, does not fit well with theories that suggest the attack comes because of a belief that high unemployment will produce monetary and fiscal expansion that attempts to create inflation and run down the Phillips curve. It does fit theories that suggest a sudden devaluation can produce a sudden reduction in the real fiscal burden of bank or corporate failures and bailouts generated by the crisis. Since this involves future primary surpluses being lower than needed to finance the increased fiscal burden without devaluation, in some sense the post-attack fiscal policy is "expansive", but it is not aimed at producing Phillips-curve inflation.

(c) Suppose we model policy-maker behavior assuming they continually update their estimated backward-looking Phillips curve, using methods that allow for time variation in the coefficients. If they assume that the constant term in the Phillips curve is likely to be changing faster, the economy is likely to stay near the Kydland-Prescott Nash equilibrium most of the time, whereas if they assume that the slope coefficient is likely to be changing faster, the economy is likely to stay fairly near the optimal ("Ramsey") point, with low inflation. Explain why it matters which coefficient is assumed to move faster.

If they assume the constant term is more likely to change, then they will interpret episodes when they try to expand, with the result that inflation rises but unemployment does not fall, as showing that the Phillips curve has shifted upward. This will cause them to expand further, until they reach the Kydland-Prescott Nash equlibrium. If they assume the slope parameter is more likely to change, then they will interpret the same type of episode as showing that the Phillips curve (as a regression of unemployment on inflation) has flattened, implying that there is little room to reduce unemployment by inflating (which is in fact true). Thus attempts to exploit the Phillips curve by inflating generate evidence that this is bad policy, preventing rapid convergence to the Nash equilibrium.