FTPL EXERCISE

In the following model, we generalize the model discussed in class by allowing an arbitrary utility function, while simplifying by eliminating money. We consider a combination of "active money" with "passive fiscal" policies, in Leeper's terminology.

The agent maximizes

$$E\left[\sum_{t=0}^{\infty} \beta^t U(C_t)\right] \tag{1}$$

subject to

$$C_t + \frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t} + Y_t \tag{2}$$

$$B_t \geq 0, \text{ all t } . \tag{3}$$

The government budget constraint is

$$\frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t} . \tag{4}$$

Suppose the government's monetary and fiscal policies are to set

$$R_t = \left(\frac{U'(C_t)}{P_t}\right)^{-\theta} \tag{5}$$

$$\tau_t = -\phi_0 + \phi_1 \frac{B_{t-1}}{P_{t-1}} \tag{6}$$

Assume $\theta > 0$, so R increases when P increases, and $\phi_0 > 0$, $\phi_1 > \beta^{-1} - 1$.

(a) Show that there is only one equilibrium in which prices, taxes and debt follow stable paths. [Hint: Log the Fisher equation and use Jensen's inequality: $\log(E[X]) \geq E[\log(X)]$ because log() is concave. Use the result to derive relations of the form $E[X_{t+1}] >= X_t$ for both the positive and negative parts of $\log(U'(C_t)/P_t)$. Then use the fact that if $E_t[X_{t+1}] \geq X_t$ for all t and X_t is bounded above for all t, then X_t converges a.s.]

(b) Show that equilibrium is nonetheless not unique.

Comment: Note that in the model of the lectures uniqueness depended sensitively on money being "essential". Here money is very much non-essential, being nonexistent.

Answer: The part of the hint about taking logs seems to have been a bad idea.

The Fisher equation is the FOC with respect to bonds, which here is

$$\frac{U'_t}{P_t} = \beta R_t E_t \left[\frac{U'_{t+1}}{P_{t+1}} \right] ,$$

where we are using the notational shortcut of setting $U'_t = U'(C_t)$. Usually with an "active" interest rate rule like this, the unique stable price level can be determined from the Fisher equation and the policy rule, without reference to fiscal policy or the government budget constraint. So we try that here. Substituting the interest rate policy rule into the Fisher equation gives us

$$\frac{U'_t}{P_t} = \left(\frac{U'_t}{P_t} \right)^{-\theta} \beta E_t \left[\frac{U'_{t+1}}{P_{t+1}} \right] .$$

Multiplying both sides by $\beta^{-1/\theta}(U'_t/P_t))^{\theta}$ delivers

$$\left(\frac{U'_t \beta^{-1/\theta}}{P_t} \right)^{1+\theta} = E_t \left[\frac{U'_{t+1} \beta^{-1/\theta}}{P_{t+1}} \right] .$$

Let $Z_t = \beta^{-1/\theta} U'_t / P_t$. Then our equation above can be rewritten as simply

$$Z_t^{1+\theta} = E_t[Z_{t+1}] .$$

Now we can proceed as we did in the FTPL model of the lectures. If $Z_0 > 1$, then with positive probability $Z_1 \geq Z_0^{1+\theta}$. Appplying this idea recursively, we then conclude that with positive probability $Z_t \geq Z_0^{t(1+\theta)}$. This means that with positive probability Z_t grows to exceed any bound. With Y_t i.i.d. and bounded, it is natural to suppose that U'_t is also i.i.d. and bounded away from zero and infinity. Then Z_t can grow arbitrarily large only if P_t grows arbitarily close to zero. A symmetric argument implies that if $Z_0 < 1$, P_t must with positive probability get arbitrarily large. So the only solution to the equation that keeps P bounded is $Z_t \equiv 1$, i.e. $P_t = U'_t \beta^{-1/\theta}$.

In the models with money, we prove that the paths that involve P_t getting arbitrarily close to zero or infinity are impossible because they will violate transversality (if P_t is shrinking) or eventually fail to satisfy the FOC's because money demand will grow too strong. (Of course, when money demand does not grow fast enough as real balances shrink, we may have multiple equilibria.) Here, though, there is no money and the fiscal policy guarantees that, whatever the path of prices, the real value of the debt will not explode up or down. Thus any of the solutions to the FOC's, including those that make $U't/P_t$ unbounded, are equilibria.

Proving that real debt is stable is easy if on the right-hand-side of the tax policy rule we put B_{t-1}/P_t or B_t/P_t. However, the problem said it was B_{t-1}/P_{t-1}. This makes the argument a bit messier. In the hope that it will make the idea clearer, I first give the argument for one of the easier cases.

Suppose the tax rule were

$$\frac{B_t}{P_t} - \phi_0 + \phi_1 \frac{B_{t-1}}{P_t} \,.$$

Then, taking E_{t-1} of the equation and using the Fisher equation, we could derive

$$E_{t-1}\left[\frac{U_t' B_t}{P_t}\right] = (\beta^{-1} - \phi_1)\frac{U_{t-1}' B_{t-1}}{P_{t-1}} + \phi_0 E_{t-1}[U_t'] \,. \tag{$*$}$$

This is a stable difference equation under the condition that $\phi_1 > \beta^{-1} - 1$. It implies that

$$E_0\left[\frac{U_t'}{P_t}\right] \to \frac{\phi_0 E[U']}{1 - \beta^{-1} + \phi_1}$$

as $t \to \infty$. (Here we've used that $C = Y$ and Y is i.i.d..) But the transversality condition for private agents is just

$$\beta^t E_0\left[\frac{U_t' B_t}{P_t}\right] \to 0 \,,$$

so the TVC is satisfied for any value of P_0. Thus any initial value of P_0 is consistent with private agent optimization, and hence with equilibrium. (This of course requires that U is concave. The budget constraint of the individual is linear in private decision variables, so clearly (weakly) convex, and thus Euler equations plus TVC are sufficient. Finally we need to assume $U' > 0$ everywhere. That, and the fact that $B = 0$ is feasible, means that the standard form of the TVC can be applied.)

With the fiscal rule of the original problem, we get in place of ($*$)

$$E_{t-1}\left[\frac{B_t U_t'}{P_t}\right] = \left(\beta^{-1} - \phi_1 E_{t-1}\left[\frac{U_t'}{U_{t-1}'}\right]\right)\frac{U_{t-1}' B_{t-1}}{P_{t-1}} + \phi_0 E_{t-1} U_t' \,,$$

or,

$$E_{t-1} W_t = \left(\beta^{-1} - \phi_1 E_{t-1}\left[\frac{U_t'}{U_{t-1}'}\right]\right) W_{t-1} + \phi_0 E U' \,,$$

where we have set $W_t = U_t' B_t / P_t$.

The coefficient in this difference equation is random. However, if we took E_{t-2} of it, the i.i.d. character of $Y_t = C_t$ would give us a constant. So it turns out that when we iterate t steps ahead, we still get a manageable formula:

$$E_0 W_t = \left(\beta^{-1} - \phi_1 E_{t-1}\left[\frac{U_1'}{U_0'}\right]\right) \Phi^{\,t-1} W_0 + \sum_{s=1}^{t} \Phi^{\,s-1} \phi_0 E U' \,.$$

Here

$$\Phi = E\left[\beta^{-1} - \phi_1\left[\frac{U_1'}{U_0'}\right]\right] .$$

Because $1/X$ is convex as a function of X for $X > 0$, $EX \cdot E[1/X] < 1$, for any positive random variable X. This lets us conclude, from our assumption that $\phi_1 > \beta^{-1} - 1$, that $\Phi < 1$. Then it is clear that the TVC will hold in this case as it did with the other policy rule for τ.