FTPL WITH MONEY

1. FTPL WITH MONEY

This model is that of Sims (1994). Agent:

$$\max_{\{C_t, M_t, B_t\}} E\left[\sum_{t=0}^{\infty} \beta^t \log C_t\right] \quad \text{s.t.}$$

$$C_t(1+\gamma f(v_t)) + \frac{M_t + B_t}{P_t} + \tau_t \leq \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} + Y_t$$

$$B_t \geq 0\,, \qquad M_t \geq 0$$

$$v_t = \frac{P_t C_t}{M_t}\,.$$

f is transactions costs as a proportion of total consumption. We assume $f'(v) \geq 0$, all $v > 0$, and $f(0) = 0$. Additional conditions on f are needed to guarantee existence and uniqueness of the equilibrium under reasonable monetary and fiscal policies.

2. GOVERNMENT

GBC: $$\frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} - \tau_t$$

Monetary policy: $$\begin{cases} M_t \equiv \bar{M} \\ R_t \equiv \bar{R} \end{cases}$$

Fiscal policy: $$\begin{cases} \tau_t = -\phi_0 + \phi_1 \frac{B_t}{P_t} \\ \tau_t \equiv \bar{\tau} \end{cases}$$

Social Resource Constraint: From private constraint and GBC.

$$C_t(1+\gamma f(v_t)) = Y_t\,.$$

3. FOC's

Assume an interior solution.

$$\partial C: \quad \frac{1}{C_t} = \lambda_t(1 + \gamma f_t + \gamma f_t' v_t)$$

$$\partial B: \quad \frac{\lambda_t}{P_t} = \beta R_t E_t \frac{\lambda_{t+1}}{P_{t+1}}$$

$$\partial M: \quad \frac{\lambda_t}{P_t}(1 - \gamma f_t' v_t^2) = \beta E_t \frac{\lambda_{t+1}}{P_{t+1}}\,.$$

The ∂B and ∂M conditions imply the "money demand" or "liquidity preference" relation

$$1 - \gamma f_t' v_t^2 = R_t^{-1}\,.$$

4. Existence and uniqueness

The ∂C and ∂M equations imply

$$\frac{1 - \gamma f_t' v_t^2}{P_t C_t(1 + \gamma f_t + \gamma f_t' v_t^2)} = \beta E_t \left[\frac{1}{P_{t+1} C_{t+1}(1 + \gamma f_{t+1} + \gamma f_{t+1}' v_{t+1}^2)} \right],$$

$$\text{or, with } Z_t = \frac{M_t}{P_t C_t(1 + \gamma f_t + \gamma f_t' v_t^2)}\,, \tag{1}$$

$$Z_t(1 - \gamma f_t' v_t^2) = \beta E_t \left[Z_{t+1} \frac{M_t}{M_{t+1}} \right]. \tag{$*$}$$

5. Case 1, $M_t = \bar{M}$

Then the M terms in the previous heading's equation in Z cancel out.

Note that Z_t is a function $g(\cdot)$ of v_t alone, and that for many (not all) "reasonable" f's we can show that

(a) $g'(v) < 0$ for all v;
(b) $g(v) \xrightarrow[v\to\infty]{} 0$ and $g(v) \xrightarrow[v\to 0]{} \infty$.

6. Conditions for existence of steady state

If the model has a steady state with constant $v_t = \bar{v}$, we will have, from $(*)$,

$$1 - \gamma f'(\bar{v})\bar{v}^2 = \beta\,. \tag{$\dagger$}$$

If $f(0) = 0$, f absolutely continuous (i.e., differentiable except at a measure-zero set of points and equal to the integral of its derivative) and $f'(v)$ is monotone near $v = 0$, then $f'(v)v^2 \to 0$ as $v \to 0$, even if $f'(v) \to \infty$ as $v \to 0$. Existence of a stationary equilibrium is therefore determined by whether $\gamma f'(v)v^2$ can be as large as $1 - \beta$.

7. Two example f's

Here and in what follows we will consider two example f's:

$$f_\ell(v) = v$$
$$f_b(v) = \frac{v}{1+v} .$$

The linear f_ℓ implies that as real balances shrink to zero relative to consumption (so $v \to \infty$), consumption goes to zero at any fixed level of Y. The bounded f_b implies that as real balances dwindle away, transactions costs approach some fixed fraction $\gamma/(1+\gamma) < 1$ of endowment.

We can see from (†) that for f_ℓ, a steady state with constant v always exists and that for f_b it exists only if $\gamma \geq 1-\beta$.

8. Uniqueness

To show this equilibrium is unique, we first show that, using f_b or f_ℓ, with any initial value of Z_t above the steady state value $\bar{Z}$, $E_t[Z_{t+s}] \to \infty$ as $t \to \infty$, while with any initial value below $\bar{Z}$, $E_t[Z_{t+s} \to 0]$ as $t \to \infty$.

Z_t is monotone decreasing in v for both these f's. (Prove this for yourself.)

$\therefore$ if $Z_t < \bar{Z}$, $E_t Z_{t+1} < \theta_t Z_t$ for some $\theta_t < 1$. This means that $P[Z_{t+1} \leq \theta_t Z_t \mid Z_t] > 0$. But then if $Z_{t+1} \leq Z_t$, we will have $E_{t+1}[Z_{t+2}] < \theta_t Z_{t+1}$ therefore that with nonzero conditional probability at $t+1$, $Z_{t+2} \leq \theta_t Z_{t+1}$, and therefore that with nonzero conditional probability at t $Z_{t+2} \leq \theta_t^2 Z_t$. Continuing this argument recursively, we will have that, with nonzero conditional probability at t, $Z_{t+s} \leq \theta_t^s Z_t$. This implies that for every $\varepsilon > 0$, there is a non-zero probability that eventually $Z_t < \varepsilon$. But $v_t \to \infty$ as $Z_t \to 0$, so With non-zero probability v_t becomes arbitrarily large if initially $Z_t < \bar{Z}$. A symmetric argument shows that if $Z_t > \bar{Z}$, v_t gets arbitrarily close to zero with non-zero probability.

9. Can we rule out equilibria with arbitrarily large or small Z's?

If we assume that $Y_t \geq \bar{Y}$ with probability one for all t, The only way we can have v_t arbitrarily close to 0 with M fixed is to have P_t arbitrarily close to zero. This implies that M_t/P_t must get arbitrarily large. Suppose the agent contemplates consuming a fraction δ of his real balances. As M/P grows larger, the current utility gain from consuming this fraction of real balances gets arbitrarily large. If the agent contemplates keeping M constant after the initial spending down of real balances, the resources available for consumption spending, $C(1+\gamma f(v))$, will remain constant after the initial period. Since the agent will assume that his own actions have no effect on future prices, and since C under this deviant decision rule will be if anything lower than on the original path, the effect on v at later dates will be to increase it by no more than the ratio $1/(1-\delta)$. But with P and $C(1+\gamma f)$ held fixed, we can calculate

$$\frac{d\log C}{d\log M} = \frac{\gamma f' v}{1+\gamma f}\,.$$

For either of our two example f's, and indeed for any concave f with $f(0) = 0$, this expression is bounded above by one. Thus the future utility costs generated by the reduction in M by the factor $1-\delta$, discounted to the current date, remain bounded, no matter how large is current M/P. But the current utility benefits of reducing M by this factor become unboundedly large as M/P increases. So it must be that the agent can increase expected utility by consuming some of his real balances if M/P gets large enough.

10. Proving downward explosion of Z is not possible

- If Z explodes downward, v approaches infinity for either of our two example f's. But referring again to $(*)$, we see that if $f' \cdot v^2$ increases with Z, $(*)$ may eventually may be impossible to satisfy, because the factor $1-\gamma f' v^2$ turns negative.
- In other words, at high enough v, the marginal value of additional real balances in this case becomes so large that no level of expected inflation, no matter how high, can leave agents satisfied not to invest in larger real balances.
- For our example $f_\ell(v) = v$, $f'v^2 = v^2$, so clearly the downward explosions are impossible.
- For $f_b(v) = v/(1+v)$, $f'v^2 = v^2/(1+v)^2$, which approaches one as $v \to \infty$. So for this case, downward explosion can be ruled out only if $\gamma > 1$.
- The conclusion is that the initial price level is unique and equal to the unique price level consistent with the unique steady-state v value for f_ℓ or for f_b with $\gamma > 1$.
- But for f_b with $\gamma < 1$, any price level above the unique steady-state v value is an equilibrium value.
- The higher initial price level corresponds to equilibria in which expected future inflation explodes and real balances shrink toward zero. In other words, they are equilibria that converge to a non-monetary, barter steady state.
- The case $0 < \gamma < 1-\beta$ is a special case. Under this condition there is no steady state. Then every initial price level is consistent with equilibrium, and all the equilibria converge to the barter steady state.

11. Government debt

- Stopping the analysis here, ignoring fiscal policy, was the mistake of the earlier literature. Dividing the government budget constraint by C_t, we arrive at

$$\frac{B_t}{P_t C_t} = R_{t-1} \frac{P_{t-1} C_{t-1}}{P_t C_t} \frac{B_{t-1}}{P_{t-1} C_{t-1}} - \frac{\tau_t}{C_t}$$

- If we substitute the policy rule for τ_t into this equation and, using the bond first order condition, take its expectation as of $t-1$, we get

$$\frac{B_t}{P_t C_t} = (\beta^{-1} - \phi_1) \frac{B_{t-1}}{P_{t-1} C_{t-1}} + \phi_0 (1 + \bar{v}) E[Y_t^{-1}] \,,$$

a stable difference equation in expected B/PY so long as $\phi_1 > \beta^{-1} - 1$.
- But if ϕ_1 is smaller than this, expected real debt (and thus with positive probability actual real debt) explodes exponentially up or down. Under the natural assumption that the government can hold at most a bounded amount of private sector real assets, the downward explosion (which would entail negative government debt) is impossible. The upward explosion might be possible if it is not too rapid, but with $\phi \leq 0$, it can be shown that such paths would violate transversality.
- Thus the unique equilibrium price level we have found with constant M does not actually correspond to an equilibrium unless the constant-M policy is acccompanied by a fiscal policy drawn from a restricted class.
- This is the point of Sargent and Wallace's "Unpleasant Monetarist Arithmetic" paper.

12. Equilibrium with R constant

- With R constant it is immediate from the liquidity preference relation that v is constant also, assuming the model has a steady state. With f_ℓ, a steady-state value for v exists for any $R > 1$. With f_b, such a v exists for any $R < 1/(1-\gamma)$. Larger values of R correspond to higher values of steady state inflation. If that is too high, people cannot be motivated to hold stable real balances with $f = f_b$.
- Knowing a unique equilibrium value for v does let us solve for a unique level of real balances $m = M/P$, but this is not enough to give us a unique initial price level.

13. Initial P with R constant

- With the $\phi_1 > \beta^{-1} - 1$ fiscal policy, the government budget constraint will make real debt follow a stable path. Though we do not check this in detail here, this case makes equilibrium non-unique with R constant.

- If instead τ is constant the government budget constraint, divided through by C_t and passed thorugh the E_{t-1} operator, becomes
$$\frac{B_t + M_t}{P_t C_t} = \beta^{-1} \frac{B_{t-1} + M_{t-1}}{P_{t-1} C_{t-1}} - \frac{R-1}{\bar{v}} - E_{t-1} \frac{\tau}{C_t} .$$
Note the appearance of a seignorage term here, because M is not constant. This is an unstable difference equation in $(B + M)/PC$. Its unique stable solution is
$$\frac{B_t + M_t}{P_t C_t} = \frac{R-1}{\beta^{-1} - 1} \left(\bar{v}^{-1} + (1 + \bar{v}) \tau E[Y_t^{-1}] \right) \equiv \bar{A}$$

14. Initial P from initial $(B + M)/P$

Now go back to the budget constraint in its original form, in the initial period:
$$\frac{B_0 + M_0}{P_0} = R \frac{B_{-1}}{P_0} + \frac{M_{-1}}{P_0} - \tau .$$
Using our unique equilibrium values $\bar{A}$ and $\bar{v}$, we can rewrite this as
$$\frac{\bar{A} Y_0}{1 + \bar{v}} = R \frac{B_{-1}}{P_0} + \frac{M_{-1}}{P_0} - \tau .$$
There is only one thing in this equation that can adjust to create equilibrium at time 0: the P_0 on the right-hand side. So there is a uniquely determined initial price level with positive initial total government liabilities, unless $RB_{-1} = M_{-1} = 0$ or $\bar{A} \leq 0$.

15. Are $\bar{A}$ and B_0 positive?

- $\bar{A}$ could be zero or negative only if $R \leq 1$, as can be seen from the definition of $\bar{A}$.
- To check for positivity of B_0, we can write
$$\bar{A} = \frac{B_0}{P_0 C_0} + \bar{v}^{-1} = \frac{R-1}{\beta^{-1} - 1} \left(\bar{v}^{-1} + (1 + \bar{v}) \tau E[Y_t^{-1}] \right)$$
- If $R = \beta^{-1}$ (non-inflationary equilibrium), positive debt is sustained if and only if $\tau > 0$.
- If $R > \beta^{-1}$, seignorage revenue allows larger real debt, indeed allows positive real debt with zero taxes.
- If $R < \beta^{-1}$, taxation is being used to retire money balances and create a positive real return on money. Positive interest-bearing debt requires an excess of taxes over the amount required to sustain the chosen rate of shrinkage in money balances (which is also the rate of deflation).

References

SIMS, C. A. (1994): "A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory*, 4, 381–99.