EXERCISE ON RANDOM LAGRANGE MULTIPLIERS AND TVC

(1) Consider the following variant of the standard LQ permanent income model, in which we use a different form of the accumulation constraint from that used in class, and we relax the condition $R\beta = 1$:

$$\max_{\{C_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t (C_t - \tfrac{1}{2} C_t^2) \right] \tag{1}$$

$$\text{subject to } C_t + A_t \le R A_{t-1} + Y_t \tag{2}$$

$$A_t \ge 0 \tag{3}$$

$$Y_t > 0 \text{ with probability one }, \; EY_t < \infty, \; Y_t \; i.i.d. \tag{4}$$

(a) Show that the objective function in this modified model is concave.
It is concave in each C_t separately, because the second derivative w.r.t. C_t is $-\beta^t < 0$. Therefore for every $\hat{C}_t$ and $\bar{C}_t$ sequence for which the sum in the definition of the objective function is convergent, we will have

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t U(\alpha \bar{C}_t + (1-\alpha)\hat{C}_t) \ge \lim_{T\to\infty} \alpha \sum_{t=0}^{T} U(\bar{C}_t) + (1-\alpha) \sum_{t=0}^{T} U(\hat{C}_t) \,,$$

which simply states that the objective function is concave on its domain.

(b) Find the Euler equations and transversality conditions.
Euler equations:

$$\partial C_t : \qquad 1 - C_t = \lambda_t$$
$$\partial A_t : \qquad \lambda_t = \beta R E_t \lambda_{t+1}$$

Transversality: For any feasible $\hat{A}_t$ process that might improve on A^*,

$$\limsup E[(1 - C_t^*)(\hat{A}_t - A_t^*)\beta^t] \le 0 \,,$$

where the $*$'s indicate the candidate optimal choices and the ˆ indicates a potential alternative choice sequence.

(c) Find the optimal decision rule, setting C_t as a function of A_{t-1} and Y_t, for the "standard" form of the model, in which we replace (3) by $E[\beta^{t/2} A_t] \to 0$ and make (1) an equality, instead of an inequality.
From the FOC's we get

$$C_t = 1 - \beta R + \beta R E_t[C_{t+1}] \,.$$

This can be solved recursively to result in

$$E_t C_{t+s} = (\beta R)^{-s} + 1 - (\beta R)^{-s} \,.$$

Substituting this expression into the budget constraint and solving that forward produces, under the assumptions that

$$E_t[R^{-s} A_{t+s}] \xrightarrow[s\to\infty]{} 0$$

and $R^2\beta > 1$,

$$A_t = \frac{C_t - 1}{R^2\beta - 1} + \frac{1 - \bar{Y}}{R - 1} \,. \qquad (*)$$

Note that if $R^2\beta < 1$*, the problem has a trivial solution: set* $C_t \equiv 1$*. That policy makes* A_t *explode upward or downward, according to the budget constraint, at the rate* R^t*. But since in this case* $R < \beta^{-1/2}$*, the* $E[A_t\beta^{t/2}] \to 0$ *constraint is not violated. So restricting attention to the case* $R^2\beta \geq 1$ *is justified. When* $R^2\beta > 1$*, The condition* $E_t[R^{-s}A_{t+s}] \to 0$*, needed for the argument above, is guaranteed by the constraint* $E[A_t\beta^{t/2}] \to 0$*. The case* $R^2\beta = 1$ *allows no solution. The Euler equations in that case imply*

$$C_t = 1 - \beta^{1/2} + \beta^{1/2} E_t C_{t+1} \,,$$

which can be solved forward (assuming C_t *does not explode faster than* $\beta^{-t/2}$*, which it can't if utility is to remain bounded) to yield*

$$C_t = 1 \,.$$

But if $C_t \equiv 1$*, the budget constraint implies that* A *grows at the rate* $\beta^{-1/2}$*, which contradicts the constraint. So no solution to the Euler equations satisfies the constraints. What this means is that given any rule for choosing a* C *path that satisfies the constraints, I can (because it will violate the Euler equations) improve on it. The agent can get arbitrarily close to* $C_t \equiv 1$ *while satisfying the constraint, but cannot actually achieve* $C_t \equiv 1$*.*

In $(*)$ *we have a relation between* A_t *and* C_t*, but since these are both choice variables at* t*, this is not yet an explicit solution. For that, we have to substitute* $(*)$ *into the budget constraint and solve for* C_t*, which results in*

$$C_t = (R - (R\beta)^{-1})A_{t-1} + (R^2\beta)^{-1} - \frac{(1-\bar{Y})(1-(R^2\beta)^{-1})}{R-1} + (1 - (R^2\beta)^{-1})Y_t \,.$$

(d) Show that your solution to the standard problem does not solve the problem in this exercise.

With some more algebra, we can solve to get A_t as a function of lagged A and current Y:

$$A_t = (R\beta)^{-1}A_{t-1} + (R^2\beta)^{-1}Y_t - (R^2\beta)^{-1}\left(1 + \frac{(\bar{Y}-1)(R^2\beta-1)}{R-1}\right) . \qquad (\dagger)$$

If $R\beta > 1$ this describes a starionary process fluctuating around a mean of $\bar{A} = (1-\bar{Y})/(R-1)$. Note that this means that at the deterministic steady state, interest on A exactly covers the gap between satiation consumption and the mean $\bar{Y}$ of income. Indeed consumption is also a stationary process, fluctuating around a mean of 1. Since C goes above 1 in this solution, it is of course possible here as in the examples discussed in class to improve on this solution by simply setting $C_t = 1$ whenever the standard solution suggests consuming above the satiation level. This will produce rapid growth in A, but in the problem we started with there was no constraint on rapid A growth.
If $R\beta < 1$, equation $(\dagger)$ is explosive. A will then become either arbitrarily large or arbitrarily small (though of course still growing at a rate $(R\beta)^{-1} < \beta^{-1/2}$). But in our original problem $A < 0$ is ruled out, and if A explodes upward we can again improve on the solution by setting $C = 1$ whenever the standard solution would suggest $C > 1$.

(2) Consider the simple "new Keynesian" model

$$\text{aggregate demand}: \quad y_t = \beta E_t y_{t+1} - \theta(r_t - E_t\pi_{t+1}) + \nu_t \qquad (5)$$

$$\text{Phillips curve}: \quad \pi_t = \delta E_t \pi_{t+1} + \gamma y_t + \varepsilon_t \qquad (6)$$

$$\text{Taylor rule}: \quad r_t = \alpha_1\pi_t + \alpha_2 y_t + \alpha_3 r_{t-1} + \zeta_t \, . \qquad (7)$$

There are no constant terms because all variables are interpreted as deviations from a steady state. Use a computer — `gensys.m` will work fine — to complete the following tasks.

(a) Check existence and uniqueness for the model with $\beta = .95$, $\theta = .5$, $\delta = .8$, $\gamma = .2$; $\alpha_1 = .11$, $\alpha_2 = .01$, $\alpha_3 = .9$.
(b) For these same parameter values, compute and plot impulse responses of r, π, and y to the three shocks ε, ν, ζ, which are all interpreted as i.i.d.
(c) Determine what range of parameter values for α_1 and α_2 are consistent with existence and uniqueness. Does the "Taylor Principle", that $\alpha_1/(1-\alpha_3)$ should exceed 1, provide a necessary and sufficient condition?

Note that, because ε and ν enter with a t subscript earlier than the date on the latest variables to appear in their equations, if you use `gensys` they have to be treated as variables in the system, appearing with a lag, and dummy equations have to be added to the system that set them equal to i.i.d. shocks.

Answer:

The aggregate demand

$$y_t = \beta E_t y_{t+1} - \theta(r_t - E_t \pi_{t+1}) + v_t$$

can be rewritten as

$$\begin{aligned} y_t &= \beta y_{t+1} + \beta \left(E_t y_{t+1} - y_{t+1}\right) - \theta r_t + \theta \pi_{t+1} + \theta \left(E_t \pi_{t+1} - \pi_{t+1}\right) + v_t \\ &= \beta y_{t+1} - \theta r_t + \theta \pi_{t+1} + v_t + \beta \eta^y_{t+1} + \theta \eta^\pi_{t+1}. \end{aligned}$$

where η^y_{t+1} and η^π_{t+1} are expectation errors of the variables in the super script. Analogously we can rewrite the Phillips curve as

$$\pi_t = \delta \pi_{t+1} + \gamma y_t + \varepsilon_t + \delta \eta^\pi_{t+1}$$

and the Taylor rule as

$$r_{t+1} = \alpha_1 \pi_{t+1} + \alpha_2 y_{t+1} + \alpha_3 r_t + \varsigma_{t+1}$$

Therefore, defining $Y_t = [y_t,\ \pi_t,\ r_t,\ v_t,\ \varepsilon_t]'$, the system can be rewritten as

$$\Gamma_0 Y_{t+1} = \Gamma_1 Y_t + \Pi \eta_{t+1} + \Psi \epsilon_{t+1}$$

where η is the vector of expectation errors, $\epsilon = [\tilde{v},\ \tilde{\varepsilon},\ \varsigma]$ with each of its component i.i.d., and

$$\Gamma_0 = \begin{bmatrix} \beta & \theta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 \\ -\alpha_2 & -\alpha_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} 1 & 0 & \theta & -1 & 0 \\ -\gamma & 1 & 0 & 0 & -1 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} -\beta & -\theta \\ 0 & -\delta \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

are the input matrices for gensys.m.
(a) The output of gensys.m, given the assumed parameters values, reads
eu =
1
1
establishing both existence and uniqueness of the solution.
(b) Using the impact and G1 matrices produced by gensys.m we can compute the impulse response functions in Figure 1, 2 and 3 (details are reported in the codes in the appendix).

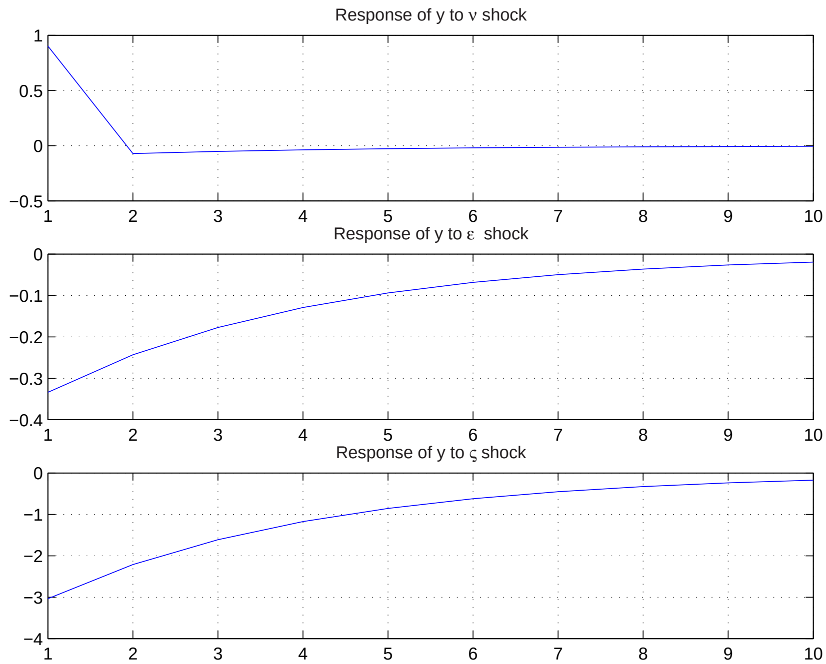

FIGURE 1

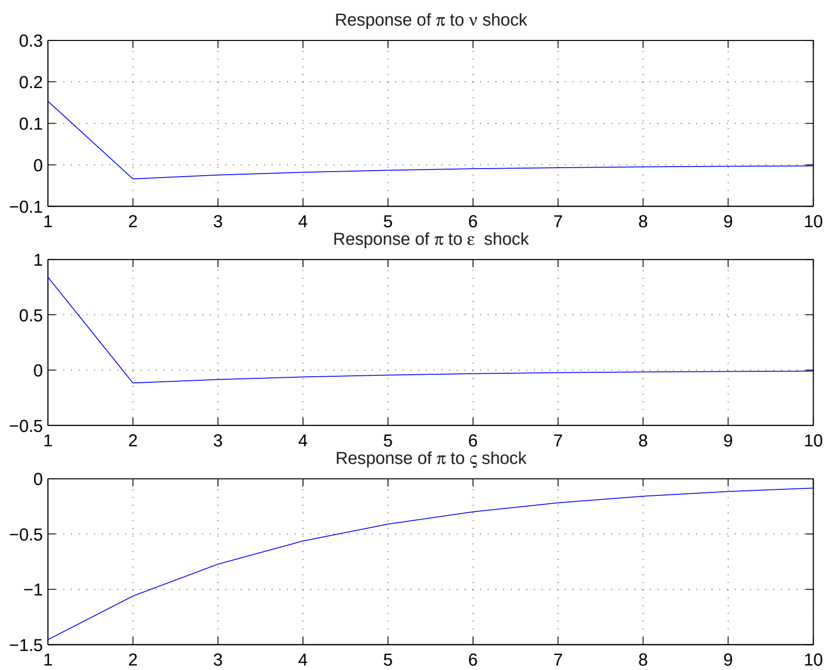

FIGURE 2

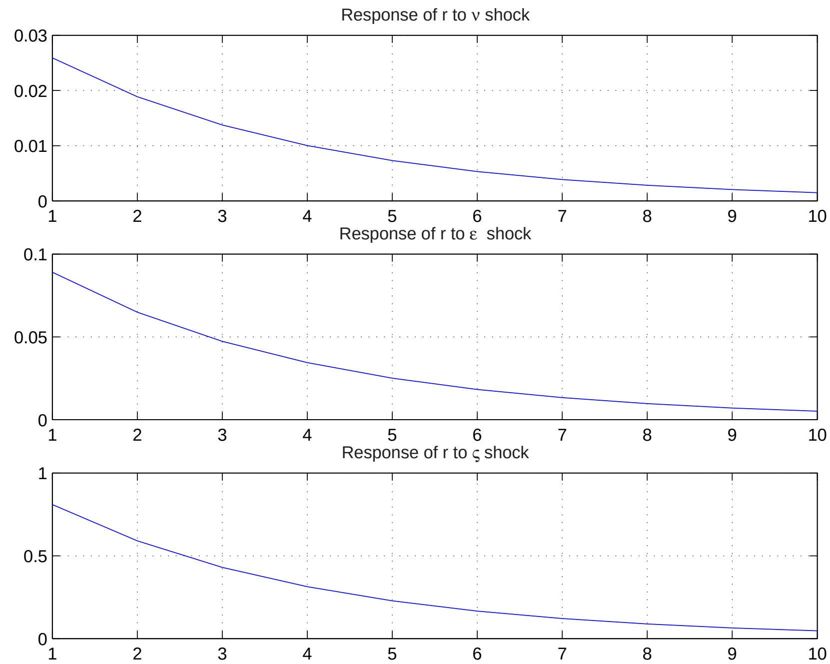

Figure 3

(c) To determine numerically what range of parameter values for α_1 and α_2 are consistent with existence and uniqueness we can make gensys.m evaluate the system over a grid of points for α_1 and α_2. In Figure 4, the shaded areas correspond to values of α_1 and α_2 such that a solution to the system exists and it's unique. The vertical line in the graph represent the minimum value α_1 that satisfies the "Taylor principle" $\alpha_1/(1-\alpha_3) > 1$. The graphs shows that the Taylor principle is neither sufficient nor necessary to deliver existence and uniqueness of the solution. Nevertheless, if we restrict our attention to the case $\alpha_2 \geq 0$, the Taylor principle is a sufficient, but not necessary, condition for uniqueness and existence.

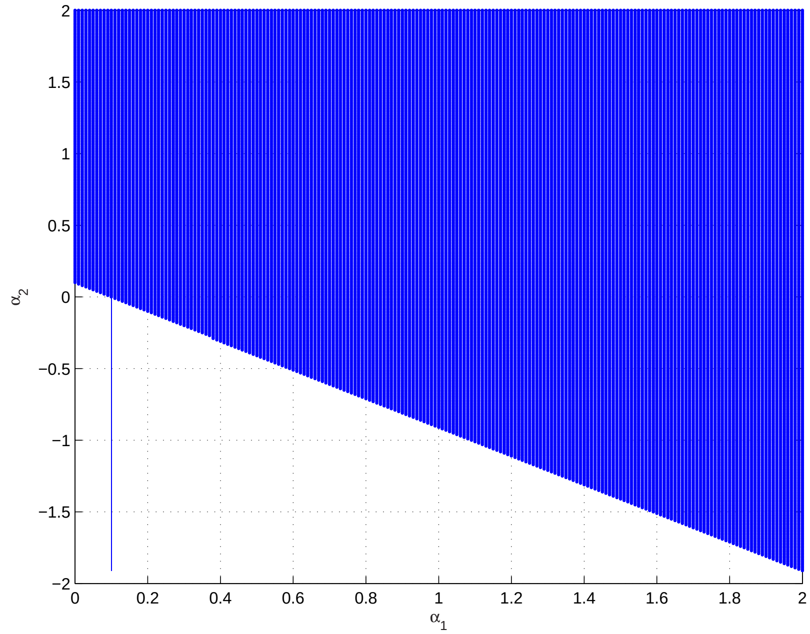

Figure 4

Appendix

All the results presented have been produced by the following matlab code:

```
beta=.95; theta=.5; delta=.8; gamma=.2; alpha1=.11; alpha2=.01 ; alpha3=.9;
pi=[-beta -theta; 0 -delta; 0 0; 0 0; 0 0]
psi=[0 0 0; 0 0 0; 0 0 1; 1 0 0;0 1 0]
g0=[beta theta 0 0 0; 0 delta 0 0 0; -alpha2 -alpha1 1 0 0; 0 0 0 1 0; 0 0 0 0 1]
g1=[1 0 theta -1 0; -gamma 1 0 0 -1; 0 0 alpha3 0 0; 0 0 0 0 0; 0 0 0 0 0]
c = [ 0; 0; 0; 0; 0];
[G1,C,impact,fmat,fwt,ywt,gev,eu]=gensys(g0,g1,c,psi,pi)

% The next lines generate the plots of the impulse-response functions.

col = impact;
for j=1:10
  resp(:,:,j)=col; % Stores the i-th response of the variables to the shocks.
  col=G1*col;   % Multiplies by G1 to give the next step response to the
                  % shocks.
end
resp1y(:,1)=squeeze(resp(1,1,:)); % "squeeze" eliminates the singleton dimensions
                        % of resp(:,:,:).  It returns a matrix with the first ten
                        % responses of the 1st variable to the 1st shock
resp2y(:,1)=squeeze(resp(1,2,:));
resp3y(:,1)=squeeze(resp(1,3,:));
resp1pi(:,1)=squeeze(resp(2,1,:));
resp2pi(:,1)=squeeze(resp(2,2,:));
resp3pi(:,1)=squeeze(resp(2,3,:));
```

```
resp1r(:,1)=squeeze(resp(3,1,:));
resp2r(:,1)=squeeze(resp(3,2,:));
resp3r(:,1)=squeeze(resp(3,3,:));
figure(1)
subplot(3,1,1)
plot(1:10,resp1y(:,1))
title('Response of y to \nu shock'); grid on
subplot(3,1,2)
plot(1:10,resp2y(:,1));
title('Response of y to \epsilon shock'); grid on
subplot(3,1,3)
plot(1:10,resp3y(:,1));
title('Response of y to \varsigma shock'); grid on
pause
figure(2)
subplot(3,1,1)
plot(1:10,resp1pi(:,1))
title('Response of \pi to \nu shock'); grid on
subplot(3,1,2)
plot(1:10,resp2pi(:,1));
title('Response of \pi to \epsilon shock'); grid on
subplot(3,1,3)
plot(1:10,resp3pi(:,1));
title('Response of \pi to \varsigma shock'); grid on
pause
figure(3)
subplot(3,1,1)
plot(1:10,resp1r(:,1))
title('Response of r to \nu shock'); grid on
subplot(3,1,2)
plot(1:10,resp2r(:,1));
title('Response of r to \epsilon shock'); grid on
subplot(3,1,3)
plot(1:10,resp3r(:,1));
title('Response of r to \varsigma shock'); grid on
pause

% The next lines search for the range of parameters alpha1 and alpha2 that
% are consistent with existence and uniqueness

a1 = [0:.01:2]; % these lines creates a grid of values for alpha1 and alpha2
a2 = [-2:.01:2];
count = 0
for i = 1:max(size(a1))
  for j = 1:max(size(a2))
    alpha1 = a1(i); alpha2=a2(j);
      g0=[beta theta 0 0 0; 0 delta 0 0 0; -alpha2 -alpha1 1 0 0; 0 0 0 1 0; 0 0 0
0 1];
      [G1,C,impact,fmat,fwt,ywt,gev,eu]=gensys(g0,g1,c,psi,pi);
      flag = eu(1)+eu(2);
      if flag ==2
        count = count +1;
```

```
        a1eu(count) =alpha1;
        a2eu(count) =alpha2;
        taylor(count) = (1-alpha3)+0.0001;
      end
    end
end
figure(4)
plot(a1eu, a2eu,taylor,a2eu); xlabel('\alpha_1'); ylabel('\alpha_2'); grid on
```