PITFALLS OF LINEAR APPROXIMATION OF STOCHASTIC MODELS

1. A LIST OF PITFALLS

- Linearized models are of course valid only locally. In stochastic economic models, this usually means valid for small stochastic disturbances that don't push us far from the non-stochastic steady state. But the restrictions this places on the validity of linearization are sometimes overdrawn. For example, linearization can be justified, for certain purposes, in non-stationary models.
- There is one apparently natural interpretation of "replacing the model with a linear-quadratic approximation" that does not lead to valid first-order expansions.
- Simulating the first-order approximation to a model and computing average welfare from the simulations does not generally lead to correct rankings of welfare across alternative policies. Higher than first order approximation is generally needed.

2. LINEARIZATION WITH NONSTATIONARITY AND UNBOUNDED DISTURBANCES

Suppose we have a dynamic system

$$x_{t+1} = f(x_t, \varepsilon_{t+1}) . \tag{1}$$

We can apply it recursively to obtain

$$x_{t+1} = f(f(x_{t-1}, \varepsilon_t), \varepsilon_{t+1}) . \tag{2}$$

Assuming the usual conditions are met, we can take a Taylor expansion of it in $\varepsilon_t, \varepsilon_{t+1}$ in the neighborhood of $\varepsilon_t = \varepsilon_{t+1} = 0$ and $x_{t-1} = x^*$ by applying the chain rule for differentiating functions of functions:

$$\begin{aligned} x_{t+1} - f(f(x^*,0),0) \doteq \\ & D_1 f(f(x^*,0),0) D_2 f(x^*,0)\varepsilon_t + D_2 f(f(x^*),0)\varepsilon_{t+1} \\ & + D_1 f(f(x^*,0),0) D_1 f(x^*,0)(x_{t-1} - x^*) . \end{aligned} \tag{3}$$

This expansion will be, like any Taylor expansion, accurate for small enough ε_t and ε_{t+1}. The error of approximation will grow small relative to the size of the ε's as the ε's get small.

If $x^* = \bar{x}$, where $\bar{x} = f(\bar{x}, 0)$ (i.e. $\bar{x}$ is a deterministic steady state value for x_t), We can apply these ideas recursively to get the s-step Taylor expansion

$$x_t - \bar{x} = \sum_{s=0}^{t-1} D_1 f(\bar{x},0)^s D_2 f(\bar{x},0)\varepsilon_{t-s} + D_1 f(\bar{x},0)^t (x_0 - \bar{x}) . \tag{4}$$

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- This formula does require $x^* = \bar{x}$. Otherwise, we would obtain a much more complicated expression involving derivatives at many different values of x. Such a "linearization about a deterministic path" can sometimes be useful, but it is not common in macroeconomics.
- The validity of the Taylor expansion for small ε and small deviations of x_0 from $\bar{x}$ does not depend on any condition on the eigenvalues of $D_1 f(\bar{x}, 0)$.
- If we want to take a limit as $t \to \infty$ and still get a valid approximation, we do require that all the eigenvalues of $D_1 f(\bar{x}, 0)$ be less than one in absolute value, as otherwise the limiting expression would not be defined. If this condition is met, and if we can uniformly bound all the ε_t's, we can make Taylor-series-approximation accuracy claims as the bound on the size of the ε's shrinks in this $t \to \infty$ case.

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- If the ε's are random and have unbounded support, e.g. i.i.d. $N(0, \sigma^2)$, we will not be able to make non-probabilistic accuracy claims. There will always be some probability that ε's are large enough to invalidate the Taylor approximation, no matter how small we make σ.
- In this case we also cannot let $t \to \infty$ and make accuracy claims. The probability that *eventually* some ε will be large enough to invalidate the approximation is one, no matter how small is λ. Once we are out of the range where the approximation is valid, the true nonlinear system might be explosive or tend toward another steady state, so the approximation would become permanently bad at that time.
- So with unbounded stochastic disturbances, our claims about the approximation take the form, "If σ is small enough, the probability that the linearized model is accurate over the time span $t = 0, \dots, T$ is greater than $1 - \delta$." And by making σ small enough, we can make T as large and δ as small as we like.
- This kind of approximation claim can be made regardless of the eigenvalues of $D_1 f$.
- Nonetheless, the eigenvalues matter. If they are all small in absolute value, the T we can obtain with a given σ and δ will grow rapidly as $\sigma \to 0$. If the largest is much bigger than one, the T will grow extremely slowly as $\sigma \to 0$, so that the approximation in practice is only good over very short time spans. Roots near one in absolute value are intermediate cases. They will allow T to grow as $\sigma \to 0$ much faster than for roots much greater than one, but much slower than for roots much less than one.

5. Two examples

Here are two examples of simulated time paths of nonlinear models and their linearizations. In one, the linearization is stable and the nonlinear model globally unstable, while in the other the linearization is unstable but the nonlinear model is globally stable. In both the

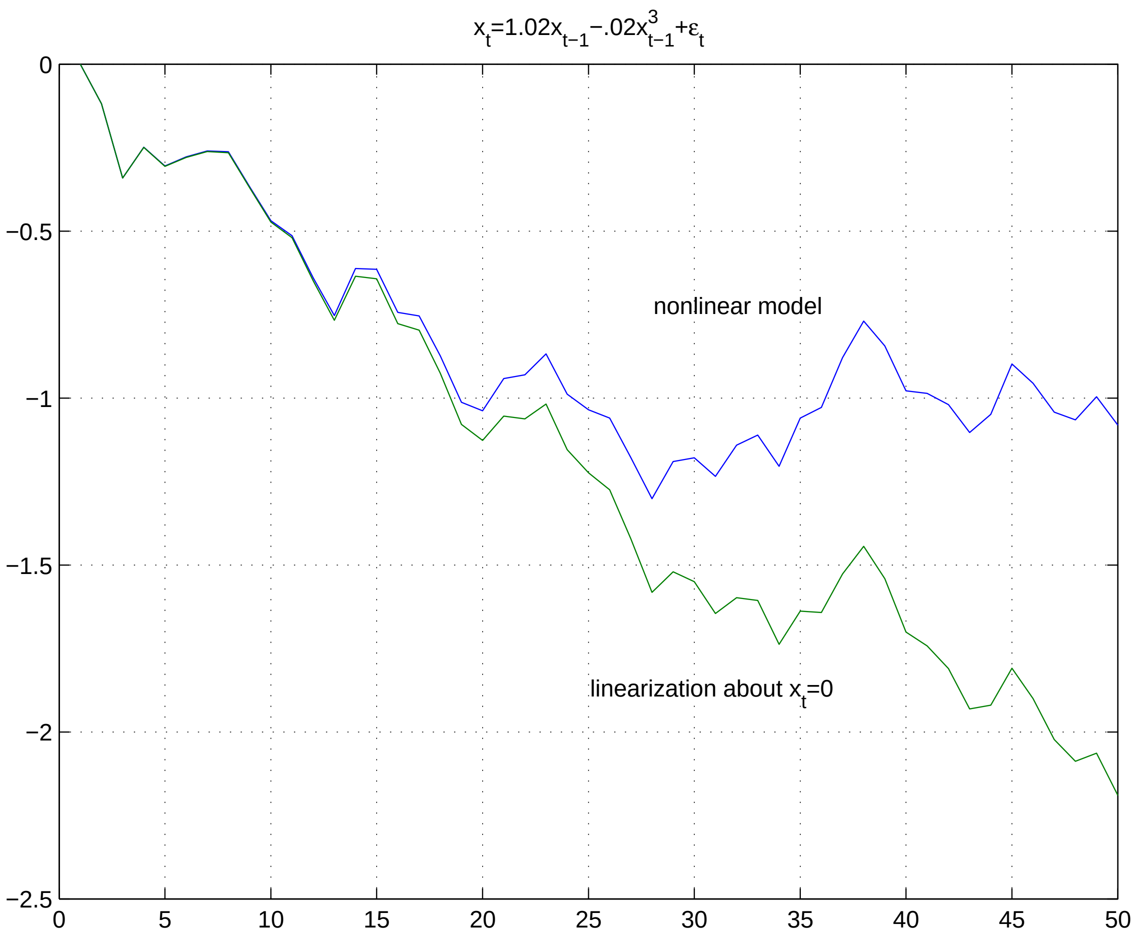
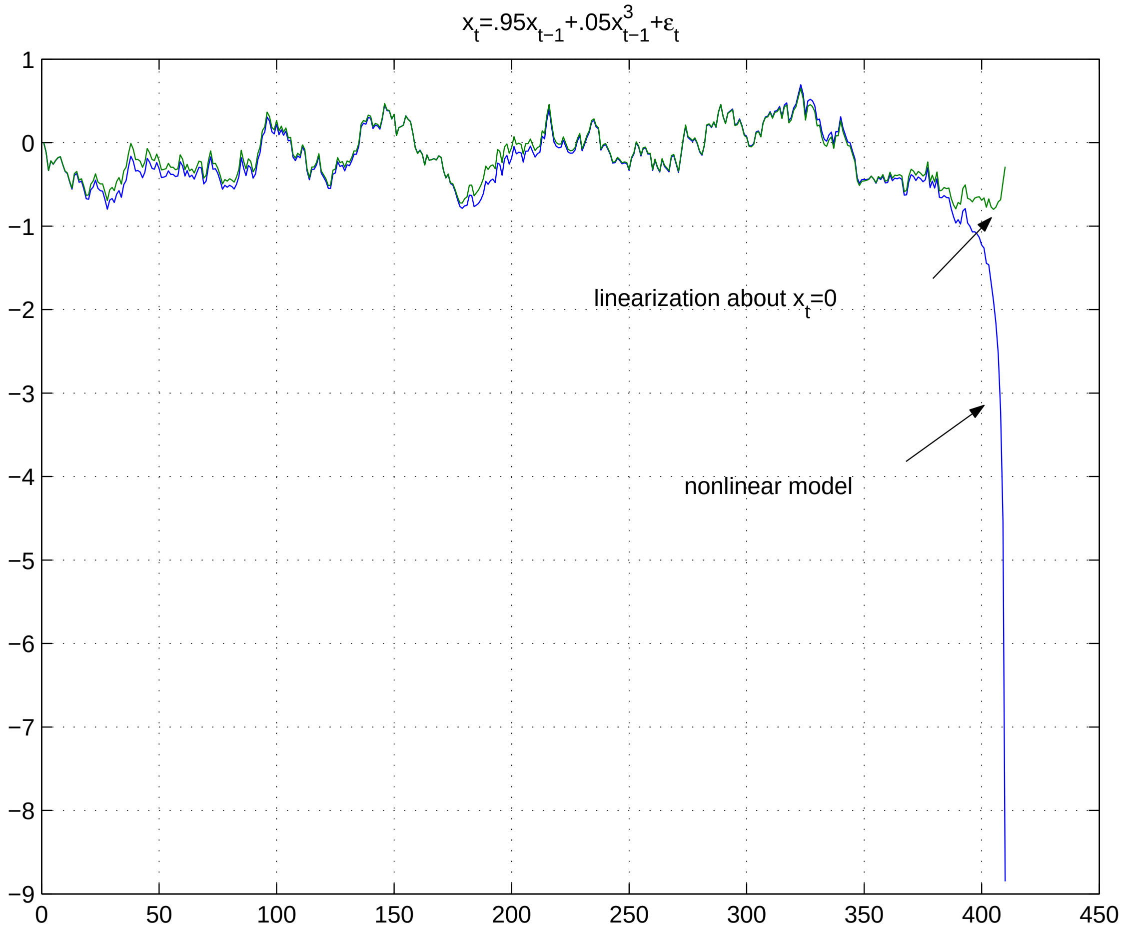


FIGURE 1

linearization is quite accurate for a time (note the very different amounts of time), but then deteriorates catastrophically.



$x_t = .95x_{t-1} + .05x^3_{t-1} + \varepsilon_t$
linearization about x_t=0
nonlinear model
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
0
50
100
150
200
250
300
350
400
450

FIGURE 2