

### CAPITAL TAX EXERCISE ANSWER

(a) From the consumer's FOC we have

$$\begin{aligned}\frac{\partial L}{\partial C} = 0 &\rightarrow \frac{1}{C_t} = \lambda_t \\ \frac{\partial L}{\partial K} = 0 &\rightarrow \lambda_t = \lambda_{t+1} \beta (1 - \tau_{t+1}) \alpha K_t^{\alpha-1} \\ &\Rightarrow C_{t+1} = C_t \beta (1 - \tau_{t+1}) \alpha K_t^{\alpha-1}\end{aligned}\tag{1}$$

and the TVC

$$\lim_{t \rightarrow \infty} \beta^t k_t = 0\tag{2}$$

where  $k_t = K_t/C_t$ . Combining the government budget constraint with the consumer's one we have the resource constraint

$$C_{t+1} + K_{t+1} = K_t^\alpha.\tag{3}$$

Combining this last equation with the Euler equation, imposing  $\tau_t = \tau$  and rearranging we have

$$\begin{aligned}k_{t+1} &= \frac{k_t}{\beta(1-\tau)\alpha} - 1 \\ &= k_0 [\beta(1-\tau)\alpha]^{-(t+1)} - \sum_{s=0}^t [\beta(1-\tau)\alpha]^{-s}\end{aligned}$$

Since  $1/\beta(1-\tau)\alpha > 1$ , we have that the only solution for  $k_t$  that does not violate the TVC is

$$k_t = k = \frac{\beta(1-\tau)\alpha}{1 - \beta(1-\tau)\alpha}.$$

(b) From the last equation we have that  $C_t = K_t k^{-1}$ . Substituting in the resource constraint we have

$$K_{t+1} (1 + k^{-1}) = K_t^\alpha.\tag{4}$$

When logged, this is a stable first order linear difference equation:

$$\log K_t = \alpha \log K_{t-1} - \log(1 + k^{-1}) = \alpha^{t+1} \log K_{-1} - \log(1 + k^{-1}) \frac{1 - \alpha^{t+1}}{1 - \alpha}.\tag{5}$$

The equation obviously has a unique steady state at

$$K_t = \bar{K} = \left( \frac{k}{1+k} \right)^{\frac{1}{1-\alpha}}.$$

(There is also an unstable steady state at  $K_t \equiv 0$ . If initial  $K$  is zero, this is the only feasible time path for  $K$ , but with any other initial condition  $K$  converges to  $\bar{K}$ .)

(c)

$$\frac{d\bar{C}}{d\tau} = [\beta(1-\tau) - 1] \alpha \beta^{\frac{\alpha}{1-\alpha}} (1-\tau)^{\frac{2\alpha}{1-\alpha}} < 0$$

(d)

$$V = \sum_{t=0}^{\infty} \beta^t \log C_t = \sum_{t=0}^{\infty} \beta^t \log k^{-1} K_t$$

Using (5),

$$\frac{dV}{d\tau} = \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{(1-\tau)[1-\beta(1-\tau)\alpha]} - \frac{1}{1-\tau} \frac{1-\alpha^{t+1}}{1-\alpha} \right)$$

$$\begin{aligned} \left. \frac{dV}{d\tau} \right|_{\tau=0} &= \sum_{t=0}^{\infty} \beta^t \left( \frac{1}{1-\alpha\beta} - \frac{1-\alpha^{t+1}}{1-\alpha} \right) \\ &= \frac{1}{1-\beta} \left( \frac{\alpha\beta - \alpha}{(1-\alpha\beta)(1-\alpha)} \right) + \frac{\alpha}{(1-\alpha\beta)(1-\alpha)} = 0. \end{aligned}$$

(e)  $B_{-1} = 0$ ,  $g_0 > 0$ ,  $\tau_t, B_t \geq 0$  and  $g_t = 0 \forall t > 0$ . Adding public debt to the problem the consumer budget constraint will become

$$C_t + K_t + B_t = (1-\tau_t) K_{t-1}^\alpha + g_t + R_t B_{t-1}$$

and the government budget constraint will be

$$B_t + \tau_t K_{t-1}^\alpha = g_t + R_t B_{t-1}. \quad (6)$$

The optimality conditions for the consumer will be given by the Euler equation (1) and the TVC (2) plus the following two conditions

$$C_{t+1} = C_t \beta R_{t+1} \Rightarrow R_{t+1} = (1-\tau_{t+1}) \alpha K_t^{\alpha-1} \quad (7)$$

$$\lim_{t \rightarrow \infty} \beta^t \frac{B_t}{C_t} = 0. \quad (8)$$

The government will maximize the consumer's lifetime utility subject to its own budget constraint plus the optimality conditions of the consumer and the consumer's budget constraint (since, given government choice of the time paths for  $\tau_t$  and  $B_t$ , the consumer will decide the time paths of consumption and savings).

Substituting out the interest rate from the problem (using equation (7)) and replacing one budget constraint into the other to obtain the resource constraints of the economy, the government problem can be written as

$$\max_{\{\tau_t, C_t, B_t, K_t\}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to equations (1), (3), (6) (with associated Lagrange multipliers  $\theta_t$ ,  $\eta_t$  and  $\mu_t$ ) (8), (2) plus  $B_{-1} = 0$ ,  $g_0 > 0$ ,  $\tau_t, B_t \geq 0$  and  $g_t = 0 \forall t > 0$ . The FOC will be:

$$\begin{aligned}\partial\tau_0 &\Rightarrow \mu_0 K_{-1} = 0 \Rightarrow \mu_0 = 0 \text{ given } K_{-1} > 0 \\ \partial B_t &\Rightarrow \mu_{t+1} \beta R_{t+1} = \mu_t \Rightarrow \mu_t = 0 \\ \partial C_t &\Rightarrow \frac{1}{C_t} + \theta_t - (1 - \tau_{t+1}) \alpha K_t^{\alpha-1} \theta_{t+1} \beta - \eta_t = 0\end{aligned}$$

$$\partial K_t \Rightarrow \theta_{t+1} \beta (1 - \tau_{t+1}) \alpha (\alpha - 1) K_t^{\alpha-2} C_t + \eta_{t+1} \beta \alpha K_t^{\alpha-1} - \eta_t = 0.$$

The last two equations imply (using equation (1)) that  $\theta_t = 0$ . Therefore the FOC coming out of  $\partial\tau_{t>0}$  will be always satisfied. We are therefore left with

$$\begin{aligned}\frac{1}{C_t} &= \eta_t \\ \eta_{t+1} \beta \alpha K_t^{\alpha-1} &= \eta_t \\ \Rightarrow C_{t+1} &= C_t \beta \alpha K_t^{\alpha-1}.\end{aligned}$$

Comparing this last equation with the constraint given by the Euler equation (1) we conclude that  $\tau_{t+1} = 0 \forall t \geq 0$ . Therefore the GBC (6) becomes

$$\begin{aligned}B_0 &= g_0 - \tau_0 K_{-1}^\alpha \\ B_t &= R_t B_{t-1} \quad \forall t > 0\end{aligned}$$

where  $R_{t+1} = \alpha K_t^{\alpha-1} \forall t > 0$ . Suppose the government decides to issue debt at time zero. We'll have

$$\lim_{t \rightarrow \infty} \beta^t \frac{B_t}{C_t} = \lim_{t \rightarrow \infty} \beta^t \frac{B_{t-1}}{C_{t-1} \beta} = \frac{B_0}{C_0} > 0,$$

violating the TVC (8). Therefore the optimal solution implies  $B_t = 0 \forall t$  and  $\tau_0 = \frac{g_0}{K_{-1}^\alpha}$  (note that this requires that  $K_{-1} \geq g_0^{\frac{1}{\alpha}}$ )

(f) This problem is analogous to the one in part (e) with the added constraint that  $\tau_0 = 0$  instead of being a control variable. The GBC will now be

$$\begin{aligned}B_0 &= g_0 \\ B_t &= R_t B_{t-1} - \tau_t K_{t-1}^\alpha \quad \forall t > 0\end{aligned}$$

Probably all that can be done with this is to write down correct FOC's and to make some qualitative statements about the nature of the solution. In any case, this part of the solution will be posted later.