FINAL EXAM

(1) (50 minutes) Consider a two-country world in which the representative agent in country A receives an endowment θ_t^X at each date t of good X, while the representative agent in country B receives an endowment of θ_t^Y of good Y. The country A agent maximizes

$$E\left[\sum_{t=0}^{\infty}\beta^t \log(X_t^A Y_t^A)\right], \tag{1.1}$$

while the agent in country B maximizes the same expression but with B superscripts on X and Y. Agents in the two countries trade the two goods. The goods are non-storable, and the price of good Y in X units at t is P_t. The two agents can also trade two assets, S^X and S^Y, which are shares in the two endowment streams.

Here are equations making this setup precise.

Budget constraint of country A agent:

$$X_t^A + \frac{Y_t^A}{P_t} - Q_t^X S_t^X + \frac{Q_t^Y S_t^Y}{P_t} \le \theta_t^X(1 - S_{t-1}^X) + \frac{\theta_t^Y S_{t-1}^Y}{P_t} - Q_t^X S_{t-1}^X + \frac{Q_t^Y S_{t-1}^Y}{P_t} \tag{1.2}$$

Budget constraint of country B agent:

$$P_t X_t^B + Y_t^B + P_t Q_t^X S_t^X - Q_t^Y S_t^Y \le \theta_t^Y(1 - S_{t-1}^Y) + \theta_t^X S_{t-1}^X P_t + P_t Q_t^X S_{t-1}^X - Q_t^Y S_{t-1}^Y \tag{1.3}$$

Market clearing:

$$X_t^A + X_t^B = \theta_t^X \tag{1.4}$$

$$Y_t^A + Y_t^B = \theta_t^Y \tag{1.5}$$

The asset market clearing conditions are implicit in the use of the same symbols for asset quantities in the two budget constraints.

(a) Define the objective function and constraints for a planner who can allocate the real endowments directly, without using markets, and who weights the utilities of the two agents equally.

The planner's problem is

$$\max_{X^A, X^B, Y^A, Y^B} \sum_{t=0}^{\infty} \beta^t \log(X_t^A X_t^B Y_t^A Y_t^B) \tag{A1}$$

subject to (1.4)-(1.5). *The original specification, though it allows asset trade, does not imply any physical possibility of moving goods between periods by storage or investment. The individual budget constraints, if the second is divided by P_t and the two are then summed, do imply an equation without assets, but that equation is implied by the two goods market clearing equations.*

(b) Solve for the planner's optimal allocation.

It is easy to see that the optimum involves equating marginal utilities, and hence actual quantities, of each good across the two agents, so the solution is

$$X_t^A = \frac{\theta_t^X}{2} = X_t^B \tag{A2}$$

$$Y_t^A = \frac{\theta_t^Y}{2} = Y_t^B \,. \tag{A3}$$

(c) Show that there is a competitive equilibrium for this economy that implements the planner's optimum. Find The Q's, P, the S's, and the X's and Y's in this equilibrium explicitly as functions of the exogenous endowments.

If there are no asset holdings and no asset trade and each country simply sells half its endowment at the market price to the other country, the resulting allocation of consumption goods certainly matches the quantity allocations in the planner's optimum. The remaining question is whether prices of the goods and of the assets can evolve in such a way that the two agents have no incentive to deviate from the planner's optimum consumption amounts and the zero asset holdings. That is, we have to check whether there are prices such that the individual FOC's are satisfied. The Euler equations for the individuals are as follows:

$$\partial X^A : \qquad \frac{1}{X_t^A} = \lambda_t^A \tag{A4}$$

$$\partial Y^A : \qquad \frac{1}{Y_t^A} = \frac{\lambda_t^A}{P_t} \tag{A5}$$

$$\partial X^B : \qquad \frac{1}{X_t^B} = P_t \lambda_t^B \tag{A6}$$

$$\partial Y^B : \qquad \frac{1}{Y_t^B} = \lambda_t^B \tag{A7}$$

$$\partial S^X(A) : \qquad Q_t^X \lambda_t^A = \beta E_t[\lambda_{t+1}^A (Q_{t+1}^X + \theta_{t+1}^X)] \tag{A8}$$

$$\partial S^Y(A) : \qquad \frac{Q_t^Y \lambda_t^A}{P_t} = \beta E_t \left[\frac{\lambda_{t+1}^A (Q_{t+1}^Y + \theta_{t+1}^Y)}{P_{t+1}} \right] \tag{A9}$$

$$\partial S^Y(B) : \qquad Q_t^Y \lambda_t^B = \beta E_t[\lambda_{t+1}^B (Q_{t+1}^Y + \theta_{t+1}^Y)] \tag{A10}$$

$$\partial S^X(B) : \qquad P_t Q_t^X \lambda_t^B = \beta E_t[\lambda_{t+1}^B P_{t+1} (Q_{t+1}^X + \theta_{t+1}^X)] \,. \tag{A11}$$

We now substitute the optimal quantity allocations into these equations to see if we can find P's and Q's that satisfy these equations. From the first four equations, we can conclude that $P_t = \theta_t^Y / \theta_t^X$ and that $P_t = \lambda_t^A / \lambda_t^B$. Using (A4) *in* (A8)*, we can arrive at*

$$\frac{Q_t^X}{\theta_t^X} = \beta E_t \left[\frac{Q_{t+1}^X + \theta_{t+1}^X}{\theta_{t+1}^X} \right] \,. \tag{A12}$$

If $\beta^t E_t[Q^X_{t+s}/\theta^X_{t+s}] \to 0$, this equation can be solved forward to yield

$$Q^X_t = \frac{\beta\theta^X_t}{1-\beta}\,. \tag{A13}$$

Equations (A7) and (A10) can be solved in the same way to derive a similar expression for Q^Y_t. It is then straightforward to use the expressions we now have in hand for P_t, Q^X_t, and Q^Y_t in the remaining two first order conditions to show that they alsoa are satisfied. This setup matches the conditions for applicability of the simple TVC. For A, w.r.t. S_X, this is

$$\beta^t E_t\left[\frac{Q^X_t(1-S^X_t)}{\theta^X_t}\right] \to 0\,. \tag{A14}$$

But since in the proposed equilibrium Q^X/θ^X and S_X are constants, this clearly holds. Checking the other TVC's is similar.
Note that, though Q^X_t/Q^Y_t does vary over time, this is only because the two asset prices are each measured in home goods units. Translated into either X or Y units using P, the two assets deliver dividends of identical value and hence have identical asset values. This implies that any equilibrium with $S^A_t \equiv S^B_t$ is equivalent to any other — the countries are issuing equal-value, equivalent securities to each other, so there is no net effect. So one could analyze an equilibrium of the type studied in class, where each country holds assets that deliver half the other country's endowment stream, and reach the same conclusion about implementability of the planner's allocation. But if one didn't see that every allocation of assets works and that the exchange of assets turns out to have no effect on budget constraints, the answer to the last parts of the question would be affected.

(d) What role does international trade in assets play in achieving this equilibrium allocation? Does this model contain a mechanism that might help explain the home bias puzzle? The Feldstein-Horioka puzzle?
There is no international trade *in assets here, so in fact it is clear that this equilibrium would obtain even if international trade in assets were impossible. One might think that, since the countries face different random endowment streams, there would be a need for asset trade to implement insurance. But since when a country gets a high endowment, the price of its product declines proportionately, the good fortune is in fact just as much the good fortune of the trading partner as of the country that received the favorable endowment shock. To the extent that one thinks that risk sharing concerns mainly country-specific endowment or technology shocks, and that a country's exports are unique, this mechanism might well explain some of the home bias puzzle. The tendency of negative supply shocks to generate improved terms of trade provides some risk sharing without asset trade.*

The Feldstein-Horioka puzzle, though, concerns why savings and investment is so highly correlated across countries. There is no investment or capital in this model, so the model cannot shed light on this puzzle.

(2) (50 minutes) The simple optimal debt and taxation model we considered in class ignored any costs of unanticipated inflation. Suppose we modify that model so that the government minimizes

$$E\left[\sum_{t=0}^{\infty}\beta^t(\tau^2+\phi\pi_t^2)\right] . \tag{2.1}$$

We are using here the non-standard notation that $\pi_t = P_{t-1}/P_t$, the *inverse* of the gross inflation rate, or the deflation rate. The government budget constraint, written in terms of real debt, is

$$b_t + \tau_t = R_{t-1}\pi_t b_{t-1} + x_t , \tag{2.2}$$

where R_t is the gross nominal interest rate on one-period debt issued at t and x_t is exogenous spending requirements. The government also faces private asset market behavior as a constraint, which in the spirit of these models we assert as if private agents were not risk averse:

$$1 = \beta R_t E_t \pi_{t+1} . \tag{2.3}$$

We assume $b_t \geq 0$ is also a constraint.

By making a late change intended to simplfy notation for this problem (writing it in terms of $\pi_t = P_{t-1}/P_t$ instead of the usual $\pi_t = P_t/P_{t-1}$) I seriously messed it up. The variance of the log of either definition of π is the same, but as stated the problem implies that there is positive value on high levels of inflation. The only deterministic steady-state solution is with infinite inflation (zero π) and infinite R, and there is no optimal solution with finite inflation rate. Near-optimum behavior will involve using debt to smooth taxes, but always generating extremely high mean inflation rates, offset by extremely high nominal interest rates. In this way π_t, the inverse of the inflation rate, can be kept arbitrarily close to zero at all times while real debt (which can remain stable because of the offsetting high values of R_t and $E_t\pi_{t+1}$) is used to stabilize the tax rate. In fact, because with very high mean inflation, tiny fluctuations in π (the inverse of inflation) can produce large proportional capital gains and losses on debt, the appearance of inflation in the objective function ends up being no constraint at all, and the usual $\tau_t \equiv \bar{\tau}$ solution can be arbitrarily closely approached.

Anyone who saw all the way through the strange structure of this problem would of course get full credit, though no one did. The initial questions about FOC's and time-inconsistency are not affected by the strange structure. The latter two parts of the question required excessive creativity for an exam, though, and the letter grading scale (but not point scores) reflects this. The most successful strategy actually used by students on the exam was to do the linearization using symbolic deterministic steady *state values ($\bar{\pi}$, $\bar{R}$, etc.). The answer below transforms the variables so that there is a finite steady state, just to show that in principle the problem has a solution.*

(a) Derive the Euler equation first-order conditions for this problem, assuming the government's plans for future actions are known by the public and that once the government has decided on its contingency plans it sticks to them. Simplify the first-order conditions by eliminating Lagrange multipliers.
The Euler equations for periods $t > 0$, assuming $b_t > 0$, are

$$\partial\tau : \qquad 2\tau_t = \lambda_t \tag{A15}$$

$$\partial b : \qquad \lambda_t = \beta R_t E_t[\pi_{t+1}\lambda_{t+1}] \tag{A16}$$

$$\partial R : \qquad \mu_t \beta E_t \pi_{t+1} = \beta b_t E_t[\lambda_{t+1}\pi_{t+1}] \tag{A17}$$

$$\partial \pi_t : \qquad 2\phi\pi_t = -\lambda_t R_{t-1} b_{t-1} + R_{t-1}\mu_{t-1} \tag{A18}$$

For $t = 0$, the $R_{t-1}\mu_{t-1}$ term in (A18) *is not present, because it arises from* (2.3)*, which is part of the constraint set only for $t \geq 0$.*
Eliminating Lagrange multipliers, we can arrive at:

$$\tau_t = \beta R_t E_t[\pi_{t+1}\tau_{t+1}] \tag{A19}$$

$$R_t^{-1}\phi\pi_{t+1} = b_t(\tau_t - \tau_{t+1}) \tag{A20}$$

(A21)

(b) Explain how time-inconsistency shows up in the first order conditions.
When we differentiate with respect to π_0, there is no applicable version of (2.3) *that contains π_0. Put another way, since the problem is being solved starting at time 0, at which point $E_{-1}[\pi_0]$ is already fixed and cannot be affected by actions of the policy-maker, the policy-maker should take no account of whether choice of π_0 violates last period's expectations. Since we assume that the policy-maker's plans now for future choices of π affect current behavior, the policy-maker does have to be concerned with expectations of his actions after $t = 0$. This is time-inconsistency: the policy maker, if allowed to re-start the solution at a later date while retaining full credibility, would choose differently than if required to stick to announced plans.*

(c) Linearize the full model (simplified Euler equations plus constraints) and put it in the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi\eta_t\,, \tag{2.4}$$

where η_t is a vector of endogenous expectational errors, ε_t are exogenous disturbances, and y_t is the vector of variables whose time paths you are solving for.
To give the problem a well-defined deterministic steady state, define $\Theta_t = R_t^{-1}$ and $\rho_{t+1} = R_t\pi_{t+1}$. These are the nominal discount factor on bonds and the ex post real return on bonds, respectively. Then the Euler equations, Fisher equation,

and budget constraint form the four-equation system:

$$\tau_t = \beta E_t[\rho_{t+1}\tau_{t+1}] \tag{A22}$$
$$\Theta_t^2\phi\rho_{t+1} = b_t(\tau_t - \tau_{t+1}) \tag{A23}$$
$$E_t\rho_{t+1} = \beta^{-1} \tag{A24}$$
$$b_t = \rho_t b_{t-1} - \tau_t + x_t\,. \tag{A25}$$

This system has a deterministic steady state for every value of steady state real debt $\bar{b} > 0$*. It satisfies*

$$\bar{\rho} = \beta^{-1} \tag{A26}$$
$$\bar{\tau} = (\beta^{-1} - 1)\bar{b} + \bar{x} \tag{A27}$$
$$\bar{\Theta} = 0\,. \tag{A28}$$

Note that this can only be interpreted as a limiting case, because it implies that one-period debt can be purchased for free at t*, but nontheless has a postive real return. The interpretation is that expected inflation is extremely high (so that* π *is near zero) and that* R *is also extremely high, but with* $R\pi$ *finite.*
Note that before we even linearize, we can multiply (A23) *by* ρ_{t+1} *and take* E_t *of it, using* (A22) *and* (A24)*to obtain*

$$\Theta_t^2\phi E_t\rho_{t+1}^2 = \beta^{-1}(\tau_t - \tau_t) = 0\,. \tag{A29}$$

Since we know that $E_t[\rho_{t+1}] = \beta^{-1} > 0$*, this equation can hold only if* $\Theta_t^2 = 0$*. So* Θ_t *is identically zero, not just zero in steady state. But then when we consider* (A23) *with* $\Theta_t \equiv 0$*, we see that it implies that, so long as* $b_t > 0$*,* τ_t *is constant. Then taking* E_{t-1} *of* (A25) *and solving forward, we find that* b_t *is the discounted present value of future primary surpluses, and if* x_t *is i.i.d.,* b_t *is constant. This then reproduces the real solution without any* $\phi\pi_t^2$ *term in the loss function. In other words, with this specification, the appearance of* π_t^2 *in the loss function makes no difference.*
But to proceed with answering the question that was asked, we can linearize (A22)-(A25) *to obtain*

$$\hat{\tau}_{t+1} + \beta\bar{\tau}\hat{\rho}_{t+1} = \hat{\tau}_t + \eta_{1t} \tag{A30}$$
$$0 = \bar{b}\hat{\tau}_t - \bar{b}\hat{\tau}_{t+1} \tag{A31}$$
$$\hat{\rho}_{t+1} = \eta_{2t} \tag{A32}$$
$$\hat{b}_t = \hat{\rho}_t\bar{b} + \beta^{-1}\hat{b}_{t-1} - \hat{\tau}_t + \hat{x}_t\,. \tag{A33}$$

We can use (A31) *to eliminate* τ *from* (A30)*, and the resulting equation is equivalent to* (A32)*, so we have a redundant equation, which is a good thing since in the linearization* Θ_t *has disappeared. Dropping* (A30) *and ordering the variables* *as (τ, ρ, b) we can cast the system into the requested canonical form by setting*

$$\Gamma_0 = \begin{bmatrix} \bar{b} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -\bar{b} & 1 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} \bar{b} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta^{-1} \end{bmatrix}, \qquad \Phi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{A34}$$

(d) It seems plausible that as $\phi \to 0$ the solution might converge to that we derived in class, in which $\tau_t = \tau_{t-1}$ whenever $b_{t-1} > 0$, and that as $\phi \to \infty$ the solution might converge to Barro's original random walk solution. Can you show that these conjectures are true? [This last part probably does not have a neat answer that can be produced in the time you have. Say what you can about it without spending disproportionate time on it.]
The first conjecture is in a sense true — we get the $\tau_t \equiv \tau_{t-1}$ regardless of the value of ϕ. So the second conjecture can't be true. This is because in the strange setup of this problem it is possible to eliminate all costs of unpredictable price changes by making the mean level of inflation high enough.

(3) (35 minutes)

(a) Explain why, in an economy governed by a natural rate Phillips curve model, a policy authority that estimates the Phillips curve by a least squares fit of a regression of unemployment on inflation and optimizes its choice of inflation each period as if its current estimates were exactly and permanently correct will converge only very slowly to the Kydland-Prescott time consistent equilibrium.
The reason for the slow convergence is that the movement toward the time-consistent equilibrium can occur only by deliberate changes in inflation chosen by the government. But every time the government makes such a change, it generates data in which inflation moves and (because the public is assumed to anticipate policy actions) there is no effect on unemployment. This makes the benefits of inflating to reduce unemployment appear low, even though the policy-makers are using an incorrect model. Another way to put it is that the faster policy makers raise the inflation rate, the stronger the evidence in the data against any good effects from increasing inflation. Surprisingly few people explained this correctly.

(b) Proposition: Since government spending has to be paid for somehow, the result that capital tax rates should optimally decline toward zero as $t \to \infty$ implies that labor tax rates must optimally rise over time to compensate for the lost revenue, if only capital and labor taxes are available and expenditure is given exogenously. Is this true or false? Explain your answer.
This proposition would only make sense if there were no government debt, so that the government budget had to balance each period. If there is government debt, we can make capital taxes decline and labor taxes be constant by using the high initial capital taxes to retire some of the government debt (or, if $b < 0$ is possible, to

acquire government assets), so that by the time the capital tax becomes negligible the constant labor tax is aligned with expenditures plus debt service.

(c) If we altered the Diamond model to allow for very long-lived (but finite-lived) agents, while preserving the overlapping generations structure, could we make its results approach those of an infinite-lived agent model, i.e. restore Ricardian equivalence, at least approximately? Explain your answer.

In an OG model Ricardian equivalence fails because debt-financed expenditures may be paid for by taxes on later generations. This allows current generations to substitute debt for real capital in their savings — i.e. for debt to "crowd out" investment, and thereby impose a real cost on future generations. This can happen whenever lives are finite, so in that sense lengthening lives will not restore Ricardian equivalence. But there are two ways to argue that Ricardian equivalence is a better approximation with longer lives. One is to note that taxes generally are fairly stable over time, so that debt issuance (that is not inflationary) will involve a permanent upward shift in the tax level. Then the longer people live, the larger a fraction of the present value of newly issued debt that will appear in their intertemporal budget constraints. The other way to make the argument is to observe that even if taxes are not changed when debt is issued, but simply postponed, there is a limit on how long they can be postponed, because the required amount of taxation increases exponentially the longer the taxes are postponed. Since there is in reality a limit to how high the tax rate can be driven, there is an upper bound to how long taxes can be postponed. If people live considerably longer than this upper bound, then most people alive at the time of a debt issue must expect to be alive when it is paid for, and again approximate Ricardian equivalence will hold.