FTPL EXERCISE

In the following model, we generalize the model discussed in class by allowing an arbitrary utility function, while simplifying by eliminating money.

The agent maximizes

$$E\left[\sum_{t=0}^{\infty}\beta^t U(C_t)\right] \tag{1}$$

subject to

$$C_t + \frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t} + Y_t \tag{2}$$

$$B_t \geq 0, \text{ all t }. \tag{3}$$

The government budget constraint is

$$\frac{B_t}{P_t} + \tau_t = \frac{R_{t-1}B_{t-1}}{P_t}. \tag{4}$$

Suppose the government's monetary and fiscal policies are to set $R_t \equiv \bar{R}$ and $\tau_t \equiv \bar{\tau}$.

- Show that there is just one equilibrium, associated with a unique price level, in which the real value of debt does not explode. [Hint: Try multiplying the government budget constraint through by $U'(C_t)$, then applying the E_{t-1} operator to it while using the Euler equation FOC's, more or less as we did in class.]
- Show that solutions to the Euler equations that allow the real value of debt to explode up or down cannot be equilibria.

The FOC's are

$$\partial C: \qquad U'(C_t) = \lambda_t$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right].$$

Substituting out λ gives us

$$\frac{U'_t}{P_t} = R_t E_t\left[\frac{U'_{t+1}}{P_{t+1}}\right]. \tag{$*$}$$

Shifting the government budget constraint forward by one period, multiplying it through by U'_{t+1}, applying the E_t operator to the whole equation, and using (**??**), *we can derive*

$$E_t\left[\frac{B_{t+1}U'_{t+1}}{P_{t+1}}\right] = \beta\frac{B_t U_t}{P_t} - \bar{\tau}E_t U'_{t+1}$$

This is an unstable difference equation. I did not make it explicit that you were to assume here, as we did in class, that Y_t is i.i.d. The private budget constraint, *combined with the GBC, lets us conclude that the social resource constraint is simply $C_t = Y_t$. So with i.i.d. Y_t, $E_t U'_{t+1}$ is a constant. With any exogenous process for Y, we can solve forward to obtain*

$$\frac{B_t U'_t}{P_t} = \bar{\tau} E_t \left[\sum_{s=1}^{\infty} \beta^t E_t U'_{t+s} \right] , \tag{†}$$

*so long as $\beta^t E_t[B_{t+s} U'_{t+s}/P_{t+s}] \to 0$ as $t \to \infty$. In the leading case (which you could have assumed) in which Y is i.i.d., (**??**) implies that BU'/P is a constant.*

So we have shown that there is only one equilibrium value at t for $B_t U'_t/P_t$, assuming that BU'/P does not explode. Since the problem stated that the condition was that B/P not explode, you might, if you were careful, have worried about this distinction. If you assume that U' is bounded, for example, B/P is bounded if and only if BU'/P is bounded. You could also have considered the private agent's TVC, which is $E[\beta^t U'_t B_t/P_t] \to 0$ (since the regularity conditions allowing the standard TVC are met). This condition is obviously met in the solution we have found, and the private agent's problem is concave, with convex constraints. This allows us to conclude that private agents see this as a unique solution to their own optimization problems, given the equilibrium price process. Furthermore, our forward solution shows that with any price process for which there is a solution to the Euler equations, agents can choose the B process so that transversality as well as the Euler equations are satisfied. Agents look forward and adjust their saving and consumption plans so that their income plus their wealth will finance future consumption and taxes. Since there is one and only one solution to the Euler equations that is the optimum for a concave/convex problem like this, only solutions with $\beta^t E[B_t U'_t/P_t] \to 0$ are equilibria.

Finally, one might ask whether we could satisfy transversality with $B/\underline{P} \to \infty$ rapidly while $U' \to 0$. This is actually possible, if $U'(\bar{C}) = 0$ at some finite $\bar{C} < EY$ — that is the utility function shows satiation in C at a level of consumption less than expected income. In this case there will be optimal paths in which agents simply plow their excess income into wealth in the form of bonds, which grows at β^{-t}, but is useless to them.