FTPL WITH MONEY

1. FTPL WITH MONEY

This model is that of Sims (1994). Agent:

$$\max_{\{C_t, M_t, B_t\}} E\left[\sum_{t=0}^{\infty} \beta^t \log C_t\right] \quad \text{s.t.}$$
$$C_t(1+\gamma f(v_t)) + \frac{M_t + B_t}{P_t} + \tau_t \leq \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} + Y_t$$
$$B_t \geq 0\,, \qquad M_t \geq 0$$
$$v_t = \frac{P_t C_t}{M_t}\,.$$

f is transactions costs as a proportion of total consumption. We assume $f'(v) \geq 0$, all $v > 0$, and $f(0) = 0$. Additional conditions on f are needed to guarantee existence and uniqueness of the equilibrium under reasonable monetary and fiscal policies.

2. GOVERNMENT

GBC: $$\frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} - \tau_t$$

Monetary policy: $$\begin{cases} M_t \equiv \bar{M} \\ R_t \equiv \bar{R} \end{cases}$$

Fiscal policy: $$\begin{cases} \tau_t = -\phi_0 + \phi_1 \frac{B_t}{P_t} \\ \tau_t \equiv \bar{\tau} \end{cases}$$

Social Resource Constraint: From private constraint and GBC.

$$C_t(1+\gamma f(v_t)) = Y_t\,.$$

3. FOC's

Assume an interior solution.

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t(1+\gamma f_t + \gamma f_t' v_t)$$

$$\partial B: \qquad \frac{\lambda_t}{P_t} = \beta R_t E_t \frac{\lambda_{t+1}}{P_{t+1}}$$

$$\partial M: \qquad \frac{\lambda_t}{P_t}(1-\gamma f_t' v_t^2) = \beta E_t \frac{\lambda_{t+1}}{P_{t+1}}\,.$$

The ∂B and ∂M conditions imply the "money demand" or "liquidity preference" relation

$$1-\gamma f_t' v_t^2 = R_t^{-1}\,.$$

4. Existence and uniqueness

The ∂C and ∂M equations imply

$$\frac{1-\gamma f_t' v_t^2}{P_t C_t (1+\gamma f_t + \gamma f_t' v_t^2)} = \beta E_t \left[\frac{1}{P_{t+1} C_{t+1}(1+\gamma f_{t+1} + \gamma f_{t+1}' v_{t+1}^2)} \right],$$

$$\text{or, with } Z_t = \frac{M_t}{P_t C_t (1+\gamma f_t + \gamma f_t' v_t^2)}\,, \tag{1}$$

$$Z_t(1-\gamma f_t' v_t^2) = \beta E_t \left[Z_{t+1} \frac{M_t}{M_{t+1}} \right]. \tag{$*$}$$

5. Case 1, $M_t = \bar{M}$

Then the M terms in the previous heading's equation in Z cancel out.

Note that Z_t is a function $g(\cdot)$ of v_t alone, and that for many (not all) "reasonable" f's we can show that

(a) $g'(v) < 0$ for all v;
(b) $g(v) \xrightarrow[v\to\infty]{} 0$ and $g(v) \xrightarrow[v\to 0]{} \infty$.

6. Conditions for existence of steady state

If the model has a steady state with constant $v_t = \bar{v}$, we will have, from $(*)$,

$$1-\gamma f'(\bar{v})\bar{v}^2 = \beta\,. \tag{$\dagger$}$$

If $f(0)=0$, f absolutely continuous (i.e., differentiable except at a measure-zero set of points and equal to the integral of its derivative) and $f'(v)$ is monotone near $v=0$, then $f'(v)v^2 \to 0$ as $v\to 0$, even if $f'(v)\to\infty$ as $v\to 0$. Existence of a stationary equilibrium is therefore determined by whether $\gamma f'(v)v^2$ can be as large as $1-\beta$.

7. TWO EXAMPLE f'S

Here and in what follows we will consider two example f's:

$$f_\ell(v) = v$$
$$f_b(v) = \frac{v}{1+v}\,.$$

The linear f_ℓ implies that as real balances shrink to zero relative to consumption (so $v \to \infty$), consumption goes to zero at any fixed level of Y. The bounded f_b implies that as real balances dwindle away, transactions costs approach some fixed fraction $\gamma/(1+\gamma) < 1$ of endowment.

We can see from (†) that for f_ℓ, a steady state with constant v always exists and that for f_b it exists only if $\gamma \geq 1-\beta$.

8. UNIQUENESS

To show this equilibrium is unique, we first show that, using f_b or f_ℓ, with any initial value of Z_t above the steady state value $\bar{Z}$, $E_t[Z_{t+s}] \to \infty$ as $t \to \infty$, while with any initial value below $\bar{Z}$, $E_t[Z_{t+s} \to 0]$ as $t \to \infty$.

Z_t is monotone decreasing in v for both these f's. (Prove this for yourself.)

$\therefore$ if $Z_t < \bar{Z}$, $E_t Z_{t+1} < \theta_t Z_t$ for some $\theta_t < 1$. This means that $P[Z_{t+1} \leq \theta_t Z_t \mid Z_t] > 0$. But then if $Z_{t+1} \leq Z_t$, we will have $E_{t+1}[Z_{t+2}] < \theta_t Z_{t+1}$ therefore that with non-zero conditional probability at $t+1$, $Z_{t+2} \leq \theta_t Z_{t+1}$, and therefore that with nonzero conditional probability at t $Z_{t+2} \leq \theta_t^2 Z_t$. Continuing this argument recursively, we will have that, with nonzero conditional probability at t, $Z_{t+s} \leq \theta_t^s Z_t$. This implies that for every $\varepsilon > 0$, there is a non-zero probability that eventually $Z_t < \varepsilon$. But $v_t \to \infty$ as $Z_t \to 0$, so With non-zero probability v_t becomes arbitrarily large if initially $Z_t < \bar{Z}$. A symmetric argument shows that if $Z_t > \bar{Z}$, v_t gets arbitrarily close to zero with non-zero probability.

9. CAN WE RULE OUT EQUILIBRIA WITH ARBITRARILY LARGE OR SMALL Z'S?

If we assume that $Y_t \geq \bar{Y}$ with probability one for all t, The only way we can have v_t arbitrarily close to 0 with M fixed is to have P_t arbitrarily close to zero. This implies that M_t/P_t must get arbitrarily large. Suppose the agent contemplates consuming a fraction δ of his real balances. As M/P grows larger, the current utility gain from consuming this fraction of real balances gets arbitrarily large. If the agent contemplates keeping M constant after the initial spending down of real balances, the resources available for consumption spending, $C(1+\gamma f(v))$, will remain constant after the initial period. Since the agent will assume that his own actions have no effect on future prices, and since C under this deviant decision rule will be if anything lower than on the original path, the effect on v at later dates will be to increase it by no more than the ratio $1/(1-\delta)$. But with P and $C(1+\gamma f)$ held fixed, we can calculate

$$\frac{d\log C}{d\log M} = \frac{\gamma f' v}{1+\gamma f}\,.$$

For either of our two example f's, and indeed for any concave f with $f(0) = 0$, this expression is bounded above by one. Thus the future utility costs generated by the reduction in M by the factor $1 - \delta$, discounted to the current date, remain bounded, no matter how large is current M/P. But the current utility benefits of reducing M by this factor become unboundedly large as M/P increases. So it must be that the agent can increase expected utility by consuming some of his real balances if M/P gets large enough.

REFERENCES

SIMS, C. A. (1994): "A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Economic Theory*, 4, 381–99.