EXERCISE: CONSUMPTION SMOOTHING

We discussed the nature of a solution to the simple linearized international borrowing and lending model we considered in class, but didn't actually lay out a solution. In this problem, you will find the actual solution. Note that the 2002 version of the course included a similar, but not identical exercise, for which an answer was posted on the course web page. This year's version involves a little more algebra, but should be easier if you start from the old solution.

The model has agent i, $i = 1, 2$ solving

$$\max_{C_i, B_i} E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_i(t)^{1-\gamma}}{1-\gamma}\right] \tag{1}$$

subject to

$$C_i(t) + B_i(t) = R_{t-1} B_i(t-1) + Y_i(t) \tag{2}$$

$$B_i(t) \geq -\bar{B}\,. \tag{3}$$

We assume that the bonds are privately issued, so that $B_1(t) = -B_2(t)$ is the market clearing condition. Assume that Y_1 and Y_2 are independent of each other, both evolving according to the same stochastic process, with mean $\bar{Y} > 0$ and satisfying

$$Y_i(t) = \rho(Y_i(t-1) - \bar{Y}) + \bar{Y} + \varepsilon_i(t)\,,$$

where $\varepsilon_i(t)$ is i.i.d. across both i and t and has mean zero.

(i) Linearize the model around a deterministic steady state in which $B = 0$, and solve for C_1, C_2, R, and B_1 as functions of the history of the exogenous processes Y_i.

The $B_1(t) = -B_2(t) = B_t = 0$ steady state produces $C_1(t) = C_2(t) = \bar{Y}$, $R_t = \beta^{-1}$. We will use the notation that for any variable X, $\hat{X}_t = (dX_1(t) + dX_2(t))/2$ and $\tilde{X}_t = (dX_1(t) - dX_2(t))/2$, where the "d" indicates deviation from steady state. The model's two budget constraints of the form (2) then can be rewritten as

$$\hat{C}_t = \hat{Y}_t \tag{A1}$$

$$\tilde{C}_t + B_t = R_{t-1} B_{t-1} + \tilde{Y}_t\,. \tag{A2}$$

The FOC's for the i'th agent are, after elimination of the Lagrange multiplier,

$$C_i(t)^{-\gamma} = \beta R_t E_t[C_i(t+1)^{-\gamma}]\,. \tag{A3}$$

Linearizing (A2) *and* (A3) *and changing to the new notation produces*

$$\tilde{C}_t + B_t = \beta^{-1} B_{t-1} + \tilde{Y}_t \tag{A4}$$

$$\tilde{C}_t = E_t \tilde{C}_{t+1} \tag{A5}$$

$$E_t \hat{C}_{t+1} = \gamma \beta \bar{Y}\, dR_t + \hat{C}_t\,. \tag{A6}$$

The solution for $\hat{C}$ *is given directly by* (A1). *For* $\tilde{C}$, *we solve* (A2) *forward. To do that, we use the fact that the process given for* $Y_i(t)$ *implies* $E_t[\hat{Y}_{t+1}] = \rho \hat{Y}_t$ *and equation* (A5), *arriving at*

$$\tilde{C}_t = (\beta^{-1} - 1)\left(B_t + \frac{\rho}{\beta^{-1} - \rho} \tilde{Y}_t\right)\,. \tag{A7}$$

For R *we use* (A1) *in* (A6) *to obtain*

$$dR_t = (\beta\gamma)^{-1}(\rho - 1)\frac{\hat{Y}_t}{\bar{Y}}\,.\,. \tag{A8}$$

For B *we use the results above in* (A2) *to obtain, after some algebra,*

$$B_t - B_{t-1} = \tilde{Y}_t \frac{1-\rho}{\beta^{-1} - \rho}\,. \tag{A9}$$

(ii) Check whether in your linearized solution B and $C_1 - C_2$ are martingales and R is i.i.d., as was true in the 2002 $\rho = 0$ version of the exercise.

That $\tilde{C}$ *is a martingale follows from the FOC* (A5) . *For* B, *we can see from* (A9) *that it is not a martingale unless* $\rho = 0$, *because* $E_t \hat{Y}_{t+1} = \rho \hat{Y}_t$.

(iii) In lecture it was asserted that the solution gets closer to autarchy as ρ increases toward one. Check whether that is true in your solution and explain how you reach your conclusion.

Autarchy is the situation where there is no international borrowing and lending, so each agent just consumes its own endowment. From (A9), *we can see that as* $\rho \to 1$, *the variation in* B *gets smaller and smaller for a given sequence of* $\tilde{Y}_t$*'s, so that in the limit* B *is a constant at its steady-state value of 0, and we are at the autarchy solution.*

It is important to note that this is not because the autarchy solution is getting closer to the complete-markets/planner's-optimum solution. As $\rho \to 1$, *the unconditional variance of* $\tilde{Y}$ *goes to* ∞, *and in the* $\rho = 1$ *limit the incomes of the two agents drift ever farther apart. So under autarchy the same thing happens to* $\tilde{C}$ *as* $\rho \to 1$. *But of course in the planner's allocation with equal weights,* $\tilde{C}_t \equiv 0$.