Econ. 504, Macroeconomic Theory Spring 2000 Chris Sims

Stochastic Lagrange Multipliers for Problems with Lagged Expectations*

We consider the problem with dynamics reduced to first order, as is used in the proofs in the notes "Random Lagrange Multipliers and Transversality". The first-order conditions we derive here generalize to higher-order models as in those notes.

The problem is

$$\max_{\{C_t,\, t=0,\dots,\infty\}} E\left[\sum_{t=0}^{\infty} \beta^t U_t\left(C_t, C_{t-1}, Z_t\right)\right] \tag{1}$$

subject to

$$g_t\left(C_t, C_{t-1}, Z_t\right) \leq 0 \tag{2}$$

$$E_t[h_{t+1}\left(C_{t+1}, C_t, Z_{t+1}\right)] \leq 0\;, \tag{3}$$

for $t = 0, \dots, \infty$. One can form a Lagrangian and derive first order conditions for this problem as follows.

Lagrangian:

$$\begin{aligned} \mathcal{L} = E\Bigg[\sum_{t=0}^{\infty} \beta^t \Big(& U(C_t, C_{t-1}, Z_t) \\ & - \lambda_t g_t(C_t, C_{t-1}, Z_t) - \mu_t h_{t+1}(C_{t+1}, C_t, Z_{t+1}) \Big)\Bigg] \end{aligned} \tag{4}$$

Euler Equation: for $t = 0, \dots, \infty$

$$\begin{aligned} & \frac{\partial U_t}{\partial C_t} + \beta E_t\left[\frac{\partial U_{t+1}}{\partial C_t}\right] = \\ & \lambda_t \frac{\partial g_t}{\partial C_t} + \beta E_t\left[\lambda_{t+1}\frac{\partial g_{t+1}}{\partial C_t}\right] + \mu_t E_t\left[\frac{\partial h_{t+1}}{\partial C_t}\right] + \beta^{-1}\mu_{t-1}\left[\frac{\partial h_t}{\partial C_t}\right] \end{aligned} \tag{5}$$

Transversality: for all feasible $\left\{\hat{C}_t\right\}_{t=0}^{\infty}$ for which the objective function is larger than that attained with $\left\{\bar{C}_t\right\}_{t=0}^{\infty}$,

$$\limsup_{t\to\infty} \beta^t E\left[\left(\frac{\partial U_t}{\partial C_t} - \lambda_t \frac{\partial g_t}{\partial C_t} - \beta^{-1}\mu_{t-1}\frac{\partial h_t}{\partial C_t}\right)\cdot\left(\hat{C}_t - \bar{C}_t\right)\right] \leq 0\,. \tag{6}$$

As in the previous notes, we leave the arguments in g_t, h_t, and U_t implicit, and in the transversality condition (6) we use $\bar{C}_t$ to denote the candidate optimum path and $\hat{C}_t$ to denote an arbitrary feasible path that improves on $\bar{C}$.[1]

The Lagrangian expectational constraints h_{t+1} enter the Lagrangian (4) with time subscript shifted forward by one relative to the non-stochastic constraints g_t, and they have Lagrange multipliers μ_t that are shifted back in time by one unit relative to the subscript on h — that is, μ_t is applied to h_{t+1}. The Euler equations and transversality condition can then be thought of as derived from derivatives of the Lagrangian exactly as in the case with no expectational constraints.

The proof that these conditions are sufficient for an optimum under concavity, convexity, and differentiability regularity conditions parallels closely that for the case with no expectational constraints. We require the same convexity and differentiability conditions on h as on g. We omit the proof here.

We also omit any attempt to give a simpler form for the transversality condition, as is possible in models without expectational constraints if they are in recursive form and the states and their marginal values are always positive. In models with expectational constraints, the division into states and controls is not so clean, and state variables for which marginal value is not of fixed sign are likely to be present.

1. Example

Consider a monetary authority solving

$$\max_{u_s,\pi_s,\, s\geq 0} -\tfrac{1}{2}E\left[\sum_{t=0}^{\infty}\beta^t(\pi_t^2+\alpha u_t^2)\right] \tag{7}$$

subject to

$$\theta_0+\varepsilon_t \leq u_{t-1}+\theta_1\pi_t \tag{8}$$

$$E_t[\pi_{t+2}] = 0\ . \tag{9}$$

Here (8) is a traditional backward-looking Phillips curve, which by itself would lead to an equilibrium with high inflation, and (9) is an inflation-targeting constraint, perhaps imposed constitutionally on the monetary authority.

To get this setup into our canonical first-order framework, we have to introduce an extra variable

$$\nu_t = E_t\pi_{t+1}\ . \tag{10}$$

[1]The fact that we ordinarily don't know how to find a feasible $\hat{C}$ that improves on $\bar{C}$ does not create any difficulty in checking this condition. Usually we first check it for arbitrary feasible C's. If we find some for which the lim sup condition might not be satisfied, we the show that these deliver a worse value of the objective function.

This definition of ν_t then becomes one of the constraints[2], and we can rewrite (9) as

$$0 = E_t[\nu_{t+1}]\,. \tag{11}$$

The Euler equations are then

$$\partial\pi: \qquad -\pi_t = -\lambda_t\theta_1 - \beta^{-1}\mu_{t-1} \tag{12}$$

$$\partial\nu: \qquad \mu_t = \beta^{-1}\psi_{t-1} \tag{13}$$

$$\partial u: \qquad -\alpha u_t = -\beta E_t[\lambda_{t+1}]\,. \tag{14}$$

The Lagrange multipliers are λ for (8), μ for (10), and ψ for (11). The transversality condition is

$$\limsup_{t\to\infty}\left\{-\beta^{t-1}\psi_{t-1}d\nu_t - \beta^t\alpha u_t du_t\right\} \le 0\,. \tag{15}$$

Stacking these equations (12)-(14) on top of the constraints (8), (10), and (11) gives us a six-equation first-order linear system in the canonical form of the paper "Solving Linear Rational Expectations Models" 2002, with the variables being π, u, ν, λ, μ, and ψ. Applying the `gensys` matlab program with $\theta_1 = .5$, $\alpha = 2$, $\beta = .95$, we find that the model meets conditions for existence and uniqueness of a stable solution. For periods $t = 1$ and later, the solution is the Phillips Curve (8) together with

$$u_t = 5 + .3448E_{t-1}\varepsilon_{t+1} + .6552E_t\varepsilon_{t+1} \tag{16}$$

$$\nu_t = 3.4483 + .726\psi_{t-1} + .6897E_t\varepsilon_{t+1} \tag{17}$$

$$\lambda_t = 20 - 4u_{t-1} - 2.1053\mu_{t-1} + 4\varepsilon_t \tag{18}$$

$$\mu_t = 1.0526\psi_{t-1} \tag{19}$$

$$\psi_t = -4.75 - .95E_t\varepsilon_{t+2}\,. \tag{20}$$

Note that (16) implies that if $E_t\varepsilon_{t+1} \equiv 0$ (the Phillips Curve disturbance is not forecastable), optimal policy simply sets $u_t \equiv 5$, the level consistent with zero expected inflation, but that when ε can be predicted, current unemployment is adjusted to partially offset the effects on inflation of next period's ε. The policy action is stronger when the Phillips curve shock is anticipated two periods in advance.

To check transversality we first note that the constraint (9) and the definition of ν (10) imply that the first term in the transversality condition (15) is zero for all $t \ge 1$. The objective function is unboundedly negative for any u_t sequence whose variance grows at β^{-t} or faster, so we know that $E[\beta^t u_t^2] \to 0$, both for $\{\bar{u}_t\}$ and for any u sequence that delivers as high a value of V. This implies that the second term in the transversality condition goes to zero.

The application of `gensys` required that we consider periods over which all six equations of the system hold, in a form where variables in the equations are all dated t or

[2] Since it is linear, both the constraint and its negative are convex, allowing us to think of the single equality constraint as two inequality constraints. In practice, we simply use a single constraint and a single Lagrange multiplier here and in (9), recognizing that the Lagrange multipliers on such exact linear constraints need not be positive.

$t-1$. Equations (10) and (11), which together form the expectational constraint, hold at $t = 0$, but at that date they involve dates $t = 0$ and $t = 1$. Shifted back in time by one, so that the latest date in the equation is $t = 0$, they would represent constraints connecting choices at $t = 0$ to expectations of them before that date, and the problem setup implies there are no such constraints.

Since u_{-1} is not chosen, but rather predetermined, (8) determines π_0. To find u_0, though, we must construct modified first-order conditions for $t = 0$. These just drop the lagged Lagrange multipliers in (12) and (13) to produce $\mu_0 = 0$, $\pi_0 = \lambda_0\theta_1$. The `gensys` equation for λ tells us, using the fact that $\mu_0 = 0$, that

$$\lambda_1 = -4u_0 + 4\varepsilon_1 + 20\,. \tag{21}$$

Using this in (14) allows us to conclude that

$$2u_0 = \beta E_0[-4u_0 + 4\varepsilon_1 + 20] \tag{22}$$

and thus that

$$u_0 = \frac{3.8E_0[\varepsilon_1] + 19}{5.8} = 3.2759 + .6552E_0\varepsilon_1\,. \tag{23}$$

Thus we see that at $t = 0$, unconstrained by previous announcements about inflation targets, the optimal policy does not pay attention to last period's expectation of next period's shock ($E_{-1}\varepsilon_1$) and chooses a lower mean unemployment rate, while responding to the expected shock for next period ($E_0\varepsilon_1$) just as it does after the first period. This first period policy is just the solution to the static unconstrained problem of maximizing $-E_0[\beta\pi_1^2 + \alpha u_0^2]$ with respect to u_0, which is natural because there is no constraint on the inflation expected at $t = 0$ to prevail at $t = 1$.

References

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1–20, `http://www.princeton.edu/~sims/`.