FINAL EXAM ANSWERS

Part II

(3) **FTPL model:** 45 points in total. Consider a model with representative agents who face the problem

$$\max_{C,B} E\left[\sum_{t=0}^{\infty}\beta^t\frac{C_t^{1-\gamma}}{1-\gamma}\right] \text{ subject to} \tag{3.1}$$

$$C_t+\frac{B_t}{Q_tP_t}=Y_t+\frac{B_{t-1}}{P_t}+\frac{B_{t-1}}{Q_tP_t}-\tau_t \tag{3.2}$$

$$B_t \geq 0\,. \tag{3.3}$$

Here C is consumption, B is the number of consols (perpetual nominal government debt) held, Q is the current nominal yield on consols. One consol pays one dollar per time period forever. So Q is one divided by the price of a consol. τ is the level of per capita lum sum taxes and P is the price level. Assume that Y_t is i.i.d. and with probability one exceeds some $\bar{Y}>0$.

The government faces the constraint

$$\frac{B_t}{Q_tP_t}=\frac{B_{t-1}}{P_t}+\frac{B_{t-1}}{Q_tP_t}-\tau_t\,. \tag{3.4}$$

Assume the government's policy is to keep $Q_t\equiv\bar{Q}$ and $\tau_t\equiv\bar{\tau}$.

(a) (7 points) Find the Euler equation first order conditions for the agents' problem.

After substituting out the Lagrange multiplier, we arrive at

$$\frac{1}{C_t^{\gamma}Q_tP_t}=\beta E_t\left[\frac{Q_t+1}{C_{t+1}^{\gamma}Q_{t+1}P_{t+1}}\right]\,.$$

(b) (7 points) Find the transversality condition(s) for the agents' problem.

The objective function is concave (so long as $\gamma>0$, as is conventionally assumed) and the constraint is linear in the choice variables. the Lagrange multiplier is $\lambda_t=C_t^{-\gamma}$, which is always positive. And $B_t=0$ is always feasible, so long as $\bar{\tau}<\bar{Y}$. Under these conditions the TVC is just the conventional

$$\beta^t E\left[\frac{B_t}{C_t^{\gamma}Q_tP_t}\right]\to 0\,.$$

The $\bar{\tau}<\bar{Y}$ condition is quite restrictive. If it does not hold, this TVC is too strong, because it assumes a zero-debt path is feasible for the individual, but the conventional condition is still a sufficient condition for optimality, even though less likely to be necessary.

(c) (7 points) Determine conditions (if any) on the parameters of the problem under which there is a deterministic steady state solution, with real variables constant and nominal variables growing at a common fixed exponential rate. The FOC and constraints imply that in such an equilibrium $C_t\equiv\bar{Y}$, $B_t/(\bar{Q}P_t)\equiv\bar{\tau}/(\beta^{-1}-1)$, and $P_{t+1}/P_t\equiv\beta(\bar{Q}+1)$.

These equations always have a feasible solution, so long as $B > 0$. If $B = 0$, then necessarily $\bar{\tau} = 0$.

(d) (12 points) Determine conditions (if any) on the parameters of the model under which there is a unique solution to the stochastic model in which the price level is determinate and real variables do not explode.

Multiplying the budget constraint by $1/C_t^\gamma$ and applying the E_{t-1} operator, we get

$$E_{t-1}X_t = E_{t-1}\left[\frac{(\bar{Q}+1)P_{t-1}C_{t-1}^\gamma}{\bar{Q}P_tC_t^\gamma}\right]X_{t-1} - E_{t-1}\frac{\bar{\tau}}{C_t^\gamma}\ , \text{ where}$$

$$X_t = \frac{B_t}{\bar{Q}P_tC_t^\gamma}\ .$$

Using the FOC and the social resource constraint $Y_t = C_t$, (obtained as usual by substituting the government budget constraint into the private constraint), this becomes

$$E_{t-1}X_t = \beta^{-1}X_{t-1} - E\left[\frac{\bar{\tau}}{Y_t^\gamma}\right]\ . \qquad (*)$$

This is an unstable difference equation in E_tX_{t+s}, whose only stable solution is

$$X_t \equiv E\left[\frac{\bar{\tau}}{(\beta^{-1}-1)Y_t^\gamma}\right] = \bar{X}\ .$$

This stable solution will always exist, since $\bar{Y} > 0$ implies $Y_t^{-\gamma} < 1/\bar{Y}^\gamma < \infty$. This will determine B_t/P_t as a function of Y_t, so long as $B_t > 0$. Then the government budget constraint will determine P_t uniquely from Y_t and quantities already fixed at t, via

$$\bar{X} = \frac{(\bar{Q}+1)P_{t-1}Y_{t-1}^\gamma}{P_tY_t^\gamma}X_{t-1} + \frac{\bar{\tau}}{Y_t^\gamma}\ .$$

This equation always has a positive solution for P_t, so long as $\bar{X} > \bar{\tau}Y_t^{-\gamma}$. Using the definition of $\bar{X}$ and some algebra, we can see that this will be true iff

$$\frac{1-\beta}{Y_t^\gamma} < \beta E\left[\frac{1}{Y_t^\gamma}\right]\ .$$

For this to be *always* true, it must hold with $\bar{Y}$ substituted for Y_t on the left. This condition is likely to be met with reasonable distributions for Y and β near 1, but it is not automatic. For any given distribution for Y, we can be sure it will not hold for β's close enough to zero.

(e) (12 points) Can explosive solutions to the Euler equations be ruled out as rational expectations competitive equilibria? Explain your answer.

The explosive solutions to the difference equation $(*)$ we derived above make X_t grow exponentially as β^{-t}, which violates the TVC. This is only suggestive, not decisive, though, since the TVC is sufficient but not in general necessary for a solution. If $\gamma < 1$ and $\bar{\tau} < \bar{Y}$, a complete argument is easy, since in that case, if B/QP gets large enough, the utility gain from eating it all up now grows without bound, while the loss from forever thereafter being forced to set $C_t = Y_t - \bar{\tau}$ has a finite discounted present value that does not grow over time. If $\gamma > 1$ or $\bar{\tau} > \bar{Y}$, the situation is quite a bit more complicated. The utility gain from eating a lot of debt now is bounded, while the utility loss from driving C close to zero at some point in the future is unbounded. No one gave a complete analysis of these cases on the exam, and this was not counted against you. Recognizing that these cases existed was "A" work.

(4) **Intergenerational tax burden shifting:** 45 points in total. In class we discussed a model where agents had a linear investment technology available and debt-financed expenditure could shift the burden of taxation onto future generations. Here we consider a version of this model in which the timing of the taxes that pay off the debt is uncertain.

Agents solve

$$\max_{C_1(t),C_2(t+1)} E_t\left[\log\bigl(C_1(t)C_2(t+1)\bigr)\right] \text{ subject to} \tag{4.1}$$

$$C_1(t) + S_t + B_t = Y_t \tag{4.2}$$

$$C_2(t+1) = \theta S_t + R_t B_t - \tau_{t+1} \tag{4.3}$$

$$B_t \geq 0\,. \tag{4.4}$$

$C_1(t)$ is consumption of generation t while young. $C_2(t+1)$ is consumption of generation t while old. S_t is physical savings by generation t. $\theta > 1$ is the rate of return to physical accumulation. B_t is government debt purchased by generation t. τ_t is lump-sum taxes paid by generation $t-1$ during the second period of life. Population is constant.

The government budget constraint is

$$B_t + \tau_t = R_{t-1}B_{t-1} + g_t\,. \tag{4.5}$$

The time path of expenditures g_t is given: $g_0 = \bar{g}$, $g_t = 0$ for $t > 0$. The financing scheme is that $\tau_t = 0$ at all dates except a single randomly chosen date t^* between 1 and T. At dates $t < t^*$, the probability of a tax next period, given information at t, is $P[\tau_{t+1} > 0 \mid \mathcal{I}_t] = 1/(T-t)$. When the tax is non-zero, it is set at $\tau_{t^*} = R_{t^*-1}B_{t^*-1}$. That is, at some random date between 1 and $T-1$, there will be a tax imposed on the old that wipes out the government's debt obligation.

(a) (20 points) Find the equilibrium time paths of C_1, C_2, and B. Determine what bounds on $\bar{g}$ and T are necessary for existence of equilibrium.

The agent's FOC's are

$$\begin{aligned}
\partial C_1 : &\qquad \frac{1}{C_1(t)} = \lambda_1(t)\\
\partial C_2 : &\qquad \frac{1}{C_2(t+1)} = \lambda_2(t+1)\\
\partial S : &\qquad \lambda_1(t) = \theta E_t \lambda_2(t+1)\\
\partial B : &\qquad \lambda_1(t) = R_t E_t \lambda_2(t+1) + \mu_t\,,
\end{aligned}$$

where $\mu_t B_t = 0$. The problem statement didn't say $S_t \geq 0$ is a constraint, but if $S_t < 0$, then in the event that the tax is in fact imposed in the second period of life, that generation will be left with negative wealth, which with log utility is implies at least $-\infty$ utility. Thus whenever $\pi_t = P[\tau_{t+1} > 0 \mid \mathcal{I}_t] > 0$ (i.e. for all $t < t^*$, under our assumptions), and whenever $B_t = 0$ (i.e. all $t \geq t^*$), we will have $S_t > 0$ as an implication of optimizing behavior.

In deriving these FOC's it is important to note that we assume as usual that this is a competitive equilibrium with lump-sum taxes, not a tax "on" government debt. That is, agents do not perceive that their individual tax obligations will increase if they buy more government debt. Indeed, since individuals see debt and private accumulation as perfect substitutes, they could well hold different ratios B/S, even though they will all, being identical and facing identical taxes, hold the same amount W_t of total savings.

$$\frac{1}{C_1(t)} = \theta E_t \frac{1}{C_2(t+1)} \,. \tag{$**$}$$

If $B_t > 0$, we must have $R_t = \theta$. (If $S_t > 0$ is a constraint, we must consider separately another case, where $B > 0$ and $S = 0$.) Let $W_t = S_t + B_t$. Let π_t be the probability at time t that the tax will be imposed at $t+1$. Then $B_0 = g_0$ and, in all subsequent periods before the tax is imposed, $B_t = g_0\theta^t$. At the date t^* when the tax is imposed, its amount will therefore be $g_0\theta^t$. So we can use $(**)$ to arrive at, for $t < t^*$,

$$\frac{1}{C_1(t)} = \theta\left(\frac{\pi_t}{\theta W_t - g_0\theta^{t+1}} + \frac{1-\pi_t}{\theta W_t}\right) \,.$$

This can be solved for $C_1(t)$ and substituted into the first-period budget constraint (4.2) to produce

$$W_t\left(2 - \frac{\pi_t g_0 \theta^t}{W_t - (1-\pi_t)g_0\theta^t}\right) = Y_t \,. \tag{$\dagger$}$$

Without the uncertainty, consumption for generations not taxed would be $Y_t/2$ in period 1 of life, $\theta Y_t/2$ in period 2 of life. Since we can now see that $W_t > Y_t/2$ when $\pi_t > 0$, necessarily $C_t < Y_t/2$. Expected welfare is necessarily lower than without the uncertainty, because the agent sees total resources as the same (Y) and investment opportunities paying the same non-random return θ as without the uncertainty, but there is some probability of having to pay the tax. W_t is bounded above by Y_t, however, because agents will never willingly drive utility to $-\infty$ by using their entire first-period income to save, leaving $C_1 = 0$. This means that it must be known in advance that $g_0\theta^{T-1} < \bar{Y}$, where $\bar{Y}$ is the lowest possible value for Y_{T-1}. Tighter bounds are possible by solving quadratic equations, but were not required. Note that, because old agents who have not been taxed have saved more than in the $\pi_t = 0$ case, they consume more than they would without uncertainty. Nonetheless their *expected* utility is lower, even though their realized utility in the second period of life may be higher.

(b) (10 points) Determine the time path of utility across generations. Is the utility loss concentrated entirely on the generation that actually pays the tax, as it is in the model with a non-stochastic tax date?

As we have already discussed, expected utility is reduced even for untaxed generations. Realized utility for the untaxed generations is also reduced. They consume less in the first period of life than they would have without the prospect of taxation, and then consume more in the second period of life than they would have without the prospect of taxation. The pair $C_1(t), C_2(t+1)$ that they choose is a pair that they could have chosen if they knew they would not be taxed, because it satisfies $C_1(t) + C_2(t+1)/\theta \le Y_t$. But without tax uncertainty they would have chosen a more even balance between first and second period consumption. Thus their realized utility is lower than it would have been without the tax uncertainty.

(c) (15 points) Does the randomness of the tax date introduce an inefficiency? That is, could we make the expected (as of time $t = 0$) utility of some generations higher, without causing any loss of utility for other generations, by using some other financing scheme?

The uncertainty is unnecessary, obviously. We could calculate the expected utility of each generation, then announce in advance a sequence of taxes that is sufficient to pay off the debt over time, yet gives each generation the same or greater expected utility. Simply stating this got nearly full credit. The random tax as specified makes the unconditional probability of paying the tax the same in every generation up to that born at $T-1$. Since the tax, if paid, is greater for the later generations, their expected welfare is lower. A constant tax therefore makes early generations worse off and is not a Pareto improvement. Simply setting each generation's tax equal to its unconditional expected tax will finance the expenditure, however. This means setting $\tau_t = g_0\theta^t/T$.

It is not true that fixing a single tax date with certainty makes everyone better off. That eliminates uncertainty, but it makes the tax-paying generation worse off, since before they faced only a probability less than one of being taxed, but now face it as a certainty. Some suggested a lottery at the initial date to determine which date would be t^*. In a formal sense this does make every generation better off in expected utility before the lottery. But since the t^*-1 generation will not be around to enjoy the pre-lottery moment during which their expected utility is higher, they will not be pleased when upon birth it is explained to them that a lottery some time in the past selected them to be taxed. A better proposal is to run a lottery at each date t, after generation t is born but before it has made its investment decisions, to determine if the tax will be imposed at $t+1$. Generation t will like this, as it gives them the chance to adapt their investments to their actual tax liability and thereby raises expected utility.

(5) **Capital and consumption taxes in an AK model:** 25 points in total. The representative agent solves

$$\max_{C,K,L,B} E\left[\sum_{t=0}^{\infty}\beta^t \log\big(C_t(1-L_t)\big)\right] \text{ subject to} \tag{5.1}$$

$$C_t(1+\nu) + K_t(1+\tau) + B_t = AK_{t-1} + L_t^{\alpha} + R_{t-1}B_{t-1} \tag{5.2}$$

$$K_t \geq 0\,, \quad B_t \geq 0\,. \tag{5.3}$$

The parameters satisfy $0 < \alpha < 1\,,\quad A > \beta^{-1} > 1$. C is consumption, L is labor input (constrained to lie between zero and one), K is capital stock, B is government debt, τ is a constant rate of capital taxation and ν is a constant rate of consumption taxation.

(a) (10 points) Show that, if initial K is positive, this model has an equilibrium in which there is steady exponential growth in C, while L shrinks toward zero. The agent's Euler equations are

$$\begin{aligned}
\partial C: &\quad \frac{1}{C_t} = \lambda_t(1+\nu) \\
\partial K: &\quad (1+\tau)\lambda_t = \beta A E_t \lambda_{t+1} \\
\partial B: &\quad \lambda_t = \beta R_t E_t[\lambda_{t+1}] \\
\partial L: &\quad \frac{1}{1-L_t} = \lambda_t \alpha L_t^{\alpha-1}\,.
\end{aligned}$$

There is no uncertainty in this model, so the E_t's can be dropped from the FOC's. Combining the C and K FOC's lets us conclude that

$$\frac{C_{t+1}}{C_t} = \frac{\beta A}{1+\tau}\,.$$

Thus a solution will have exponential growth in C at the rate $\beta A/(1+\tau)$. The L FOC then tells us that

$$\frac{1}{L_t^{1-\alpha}}(1-L_t) = \frac{\alpha(1+\nu)}{C_t} .$$

Since the left-hand side of this expression is monotone increasing in L, while the right-hand side is monotone decreasing in C, The exponential growth in C will induce decreasing L, and indeed L will have to go to zero to maintain the equality as $C \to \infty$. We will use $h(C_t)$ for the mapping from C to L implied by this equation.

The problem statement should have specified the budget constraint, so you would know whether there is government expenditure. A natural assumption is that there is no government spending, so taxes simply back the debt. Then the SRC can be written, using the exponential growth of C_t, as

$$C_0\left(\frac{\beta A}{1+\tau}\right)^t + K_t = AK_{t-1} + h\left(C_0\left(\frac{\beta A}{1+\tau}\right)^t\right) .$$

This can be solved forward to give a mapping from C_0 to K_0. Showing that this is a one-one mapping would require some algebra, which we skip. But if this is accepted, then the relation determines C_0. The uniqueness depends on our ruling out $\lim_{t\to\infty} K_t A^{-t} > 0$. This does violate transversality, and we can directly rule it out, since if K/C grows without bound, eventually we can consume part of K, never decrease C's at later dates, and yet not violate any constraints.

(b) (5 points) Show that if the capital tax rate is positive, the C growth rate is reduced, while there is no such effect from a positive consumption tax rate.

We have already shown this.

(c) (10 points) Does the fact that the capital tax reduces the growth rate mean that if both tax rates can be set freely to finance a given expenditure stream, but each must be kept constant, it is optimal to set the capital tax to zero? [You should be able to answer this qualitatively by looking at what distorting "wedges", if any, each tax introduces, without grinding out the algebra to get expressions for the actual welfare losses.]

No. The consumption tax does not distort intertemporal decisions, but it does distort the labor-leisure tradeoff. As the capital tax is increased from zero, there are increases in current consumption and decreases in future consumption along an efficient frontier, so deadweight losses are second-order, just as for the consumption tax. Thus it is likely that it is optimal to have both taxes non-zero.

(6) **Short answers:** 20 points in total.

(a) (10 points) Suppose there is an unanticipated disaster that requires a big temporary increase in government spending. Barro's analysis of optimal fiscal policy suggests that this should produce an increase in taxes, while models like that of Lucas and Stokey and some "fiscal theory of the price level" models imply that optimal policy would make little or no change in taxes in response to the disaster. What explains this difference in conclusions?

Barro does not consider the possibility that nominal government debt is in fact state-contingent. Thus his model does not give the government the option of a surprise inflation to efficiently finance a fiscal emergency. Because such surprise inflation taxes are non-distorting, it is optimal in flex-price models to use them to finance surprise deteriorations in the fiscal position.

(b) (10 points) The usual analysis of a natural rate Phillips curve, as set out by Kydland and Prescott, implies that a government that cannot commit will end up at an equilibrium with high inflation. In the Sargent/Arifovic experiments, "policy makers" and the "public" tend instead to converge to near-optimal levels of inflation. Does the Barro-Gordon analysis suggest an explanation for this? Why or why not?

Most of the Sargent-Arifovic experiments seemed to show policy-makers acting as if they were confident that they could induce favorable inflation expectations by acting predictably and smoothly to bring inflation down, and they seemed to be right about this. This could represent a Barro-Gordon sort of game theoretic equilibrium, though it is hard to determine this from the experimental data itself. The Barro-Gordon reasoning requires knowledge by each agent of what would happen in the off-equilibrium paths that are not observed. Here we would have to know that the policy makers had an idea that they would produce difficult-to-manage inflation expectations if they attempted to surprise the public, while the public would have to believe that persistent low current inflation implies low future inflation. Since the participants were not communicating, they might instead have been using simple extrapolative rules of thumb to form expectations. Sargent and Arifovic point out, though, that if the policy makers had known the "public"'s expectation formation mechanism, they would have moved inflation down more quickly. That they moved slowly may have had to do with Barro-Gordon style calculations of the possible evil consequences for expectations of behaving "erratically". It is worth noting that the Barro-Gordon game theoretic approach, though it also works off beliefs by policy makers about the effects of their actions on public expectations, is not the same as the approach that postulates adaptive expectations on the part of the public. Adaptive expectations are generally an *incorrect* model of policy behavior that may nonetheless lead to a self-confirming equilibrium, in which the public sees no further evidence that they are mistaken. The Barro-Gordeo equilibrium assumes that both the public and the policy authority are correct in their beliefs both about what happens on the equilibrium path and what would happen off of it. Of course in an actual equilibrium we see only the equilibrium path and therefore can't check agents' beliefs about about off-equilibrium contingencies, except perhaps by asking them.