# OLG EXERCISES

(1) Suppose in our overlapping generations model the utility function is

$$U\big(C_1(t), C_2(t+1)\big) = \log\big(C_1(t) \cdot C_2(t+1)\big) . \tag{A}$$

Suppose also that instead of being endowed with $\bar{Y}$ when young and nothing when old, each generation is endowed with $\bar{Y}/2$ when young and the same amount when old. Suppose the population is constant and $\theta > 1$.

(a) Show that in this case the policy of financing initial-period government spending $g_0$ entirely with debt issue and taxing only at some later date $T > 0$ will not work.

*Actually, I was wrong. It is still possible to use pure debt finance for a small enough $g_0$. It is a characteristic of this log utility function that agents always split the present value of their endowment equally between current and later period consumptions spending. That is, the fraction of the present value of their endowment that they set aside in the first period of life is invariant to the return on investment. To see this formally, note that the FOC's give us*

$$\frac{C_2(t+1)}{C_1(t)} = \theta .$$

*Then from the two budget constraints, solved to eliminate $B_t + S_t$ (which can be done if both $B > 0$ and $S > 0$, because then $R_t = \theta$), we get*

$$C_1(t) + \frac{C_2(t+1) - .5\bar{Y}}{\theta} = .5\bar{Y} - \tau_t \text{ , and therefore}$$

$$2C_1(t) = \bar{Y}\frac{\theta+1}{2\theta} - \tau_t .$$

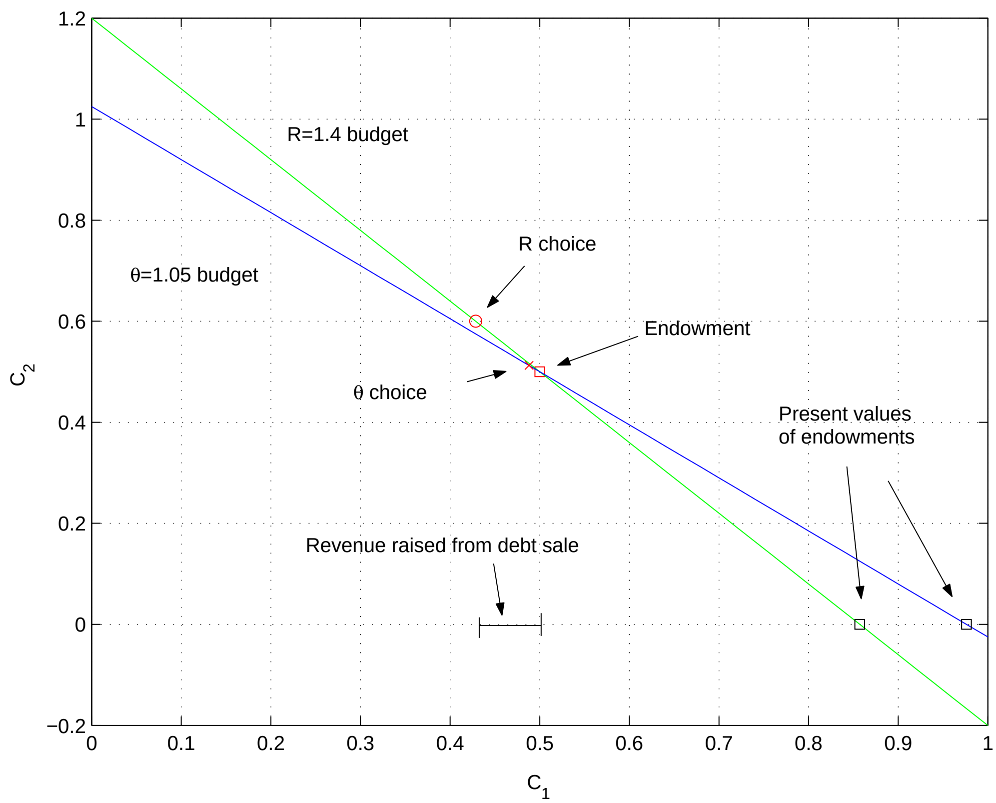
*In other words, first-period consumption is always half the present value of the endowment, after taxes. With $\theta > 1$, the present value of the endowment is less than $\bar{Y}$, so with $\tau_t = 0$ desired consumption falls below period-1 endowment by*

$$\frac{\bar{Y}}{2} - C_1(t) = \frac{\bar{Y}}{2}\left(1 - \frac{1+\theta}{2\theta}\right) = \frac{\theta-1}{4\theta}\bar{Y} .$$

*This gives the amount of savings in each generation, which is also the upper limit on the amount of debt paying $R = \theta$ that a generation will freely acquire and thus also the upper limit on the size of $g_0$ consistent with $R = \theta$ and zero taxes in the first period. This limit is zero only if $\theta = 1$.*

*However, by increasing $R$ above $\theta$, the government* reduces *the present value of the endowment, even though it strictly expands the achievable budget set. As shown in the diagram below, the budget set rotates clockwise around the endowment as $R$ increases, lowering the present value (its intersection with the $C_1$ axis). Since none of the budget set to the right of $C_1 = .5$ is attainable (because savings has to be positive), the attainable budget set strictly expands when $R > \theta$. The higher rate of return therefore induces agents to reduce $C_1$ further. There* *is a limit on this, because the total per capita resources in the next period, other than what is in the second-period endowment of the current young, are $\bar{Y}/2$. The government therefore can promise only $R \leq \bar{Y}/(2g_0)$. An $R$ higher than that would require that the holders of the debt be delivered more resources than exist in the economy. At this upper bound on $R$, the amount of resources that can be raised through debt issue is at most $\bar{Y}/6$ per capita, and this would require taxing away the entire first-period endowment of the next generation, making their utility $-\infty$. The algebra:*

$$g_0 = \frac{\bar{Y}}{2} - C_1 = \frac{\bar{Y}}{2} - \frac{1}{2}\left(\frac{\bar{Y}}{2} + \frac{\bar{Y}}{2R}\right)$$
$$= \frac{\bar{Y}(R-1)}{4R} \leq \frac{\bar{Y}}{4}\left(1 - \frac{2g_0}{\bar{Y}}\right), \therefore$$
$$g_0 \leq \frac{\bar{Y}}{6}.$$




(b) Suppose that debt and taxes are constrained to be non-negative, but that the old as well as the young can be taxed. Find a sequence of taxes and debt issue that finance the initial $g_0$ by imposing the burden only on generation $T > 0$, without imposing any burden on generations before or after $T$.

*Of course for small enough $g_0$ and $T$ this can be done as in the in-class example, by imposing no taxes until $T$ and financing with debt. If $g_0$ is bigger than can be financed with pure debt issue, to get the additional resources we will have to reduce the consumption of the first-period young or the first-period old. If we reduce the consumption of the old, their welfare is reduced. If we* *reduce the consumption of the young, their welfare must be lower* unless they are reducing consumption because they perceive a higher rate of return than $\theta$. *So the method will be to offer government debt that completely crowds out private saving by offering a higher return than $\theta$. The higher rate of return will be seen by the young as an outward shift in their budget sets, so to keep their welfare unchanged, we will have to announce that they will be taxed when old. (The problem could have been interpreted as requiring only that no generation other than generation $T$ be made* worse *off, instead of requiring that all generations other than $T$ have unchanged welfare. However by using taxes to keep the welfare of all generations other than $T$ the same, we can achieve the largest values of $T$ and finance the largest possible $g_0$.)*

*The budget constraints of an agent at t, in a period where all savings are in the form of government debt, are*

$$C_1(t) + B_t = \frac{1}{2}\bar{Y}$$

$$C_2(t+1) = \frac{1}{2}\bar{Y} + R_t B_t - \tau_{t+1} \, .$$

*The government budget constraint is*

$$B_t = \begin{cases} R_{t-1}B_{t-1} - \tau_t & t > 0 \\ g_0 & t = 0 \end{cases} \, .$$

*From these equations we can derive the social resource constraint*

$$C_1(t) + C_2(t) = \bar{Y} \, . \qquad \text{(SRC)}$$

*Optimization, together with the logarithmic form of the utility function, implies that*

$$C_1(t) = R_t C_2(t+1) \qquad \text{(C Growth)}$$

*at every date. In order to keep utility constant across generations before the date of debt retirement, then, we must have for these generations*

$$\log(C_1(t)C_2(t+1)) = 2\log(C_1(t)) + \log(R_t) = 2\log(\bar{C}_1) + \log(\theta) \, . \qquad \text{(Constant Utility)}$$

*Here $\bar{C}_1$ is the first-period consumption of a generation that faces no taxes and saves with a return $\theta$.*

*The constant-utility condition is easily seen to imply*

$$C_1(t)^2 R_t = \bar{C}_1^2 \theta \, . \qquad \text{(C ratio)}$$

*Substituting this in the $t = 0$ revenue requirement (GBC) condition then lets us solve for $R_0$, as*

$$R_0 = \theta \left( \frac{(1+\theta)\bar{Y}}{2\theta(\bar{Y} - 2g_0)} \right)^2 \, .$$

*Of course $C_1(0) = \frac{1}{2}\bar{Y} - g_0$. The three equations (C Ratio), (C growth) and (SRC) above constitute a set of three difference equations in $R$, $C_2(t)$ and $C_1(t)$, with only $R$ and $C_1$ appearing with a lag. Hence they can be solved recursively forward in time from the initial values we have found for $R_0$ and $C_1(0)$. It can be shown that the equations imply steady decrease in $C_1(t)$, and thus steady increase in $R_t$, from any initial condition, and that indeed the equations can imply positive* *solutions for $R$ and $C_1$ only over a finite span of time. This is because eventually debt expands to exceed available resources.*

*$C_1(t)$ itself is half the present value of the agent's endowment less taxes, i.e.*

$$\frac{R_t+1}{4R_t}\bar{Y} - \frac{\tau_{t+1}}{2R_t},$$

*where $\tau_{t+1}$ is the tax imposed at $t+1$ on those old at that time. This equation lets us solve for $\tau_t$ from our solution for the $R$ time series.*

(2) Consider an economy that begins at time $t = 0$ with an old generation that owns a fixed quantity $R_0$ per capita of a natural resource and a young generation that has a production technology. The natural resource can be stored without cost from one period to the next, or it can be used in the production technology to produce the consumption good. The old can sell the natural resource to the young. The problem for generation $t$ is therefore

$$\max_{C_1(t),C_2(t+1),R_t,R_{t+1}} U\big(C_1(t), C_2(t+1)\big) \tag{B}$$

subject to

$$C_1(t) + P_t R_t = f(R_t - R_{t+1}) \tag{C}$$
$$C_2(t+1) = P_{t+1}R_{t+1} \tag{D}$$
$$R_t \geq R_{t+1} \geq 0\,. \tag{E}$$

(a) Assuming the functional forms $U(x, y) = \log(xy)$ and $f(r) = r^\alpha$, with $\alpha < 1$, find the competitive equilibrium time paths of $R$, $P$, $C_1$ and $C_2$, as well as of generational utilities.

*The FOC's are*

$$\begin{aligned}
&\partial C: && \frac{1}{C_1(t)} = \lambda_1(t) \qquad \frac{1}{C_2(t+1)} = \lambda_2(t+1)\\
&\partial R_t: && \lambda_1(t)\cdot\big(P_t - \alpha(R_t - R_{t+1})^{\alpha-1}\big) = 0\\
&\partial R_{t+1}: && \lambda_1(t)\alpha\cdot(R_t - R_{t+1})^{\alpha-1} = \lambda_2(t+1)P_{t+1}\,.
\end{aligned}$$

*We can reduce these to*

$$\frac{P_t}{C_1(t)} = \frac{P_{t+1}}{C_2(t+1)} \tag{PCratio}$$
$$P_t = \alpha\cdot(R_t - R_{t+1})^{\alpha-1}\,.$$

*If we use these two equations, together with the budget constraints, to eliminate $P$, $C_1$, and $C_2$, we end up with the following equation in $R$:*

$$R_{t+1} = \frac{1-\alpha}{1+\alpha}R_t\,. \tag{R}$$

*Using the fact that initial $R_0$ is given, we therefore have a full equilibrium path for $R$, in which it declines exponentially toward zero. Letting $A = (1-\alpha)/(1+\alpha)$,* *we can then easily get*

$$P_t = \alpha R_0 A^{(\alpha-1)t}\left(\frac{2\alpha}{1+\alpha}\right)^{\alpha-1}$$
$$C_2(t+1) = \alpha A^{\alpha t} R_0^{\alpha}(1-A)^{\alpha-1}$$
$$C_1(t) = A^{1-\alpha} C_2(t)$$
$$U\big(C_1(t), C_2(t+1)\big) = 2\alpha t \log A + 2\log(\alpha R_0^{\alpha}(1-A)^{\alpha-1}) + (1-\alpha)\log A\,.$$

*So utility declines linearly over time.*

(b) Suppose we are interested in finding a fiscal policy that will result in a less rapid decline in utilities across generation. Here are two proposed policies:

- A tax on the old, proportional to their holdings of $R$ carried over from the first period of life, with the proceeds distributed as a lump sum transfer to the young;
- A tax on the young, proportional to their usage of natural resources (i.e. to $R_t - R_{t+1}$), with the proceeds distributed as a lump sum to the old.

Would either policy succeed in reducing the rate of decline of utility? Would either be Pareto-efficient?

*The first tax would change the agent's budget constraints and add a government budget constraint as follows:*

$$C_1(t) + P_t R_t = (R_t - R_{t+1})^{\alpha} + T_t$$
$$C_2(t+1) = (1-\tau)P_{t+1}R_{t+1}$$
$$T_t = \tau P_t R_t\,.$$

*Only the $\partial R_{t+1}$ FOC changes, to*

$$\frac{P_t}{C_1(t)} = \frac{P_{t+1}(1-\tau)}{C_2(t+1)}\,. \tag{PCratio$'$}$$

*This in turn leads to a changed version of the dynamics for $R$, so* (R) *becomes*

$$R_t = \frac{1-\alpha(1-\tau)}{\frac{\alpha}{1-\tau}+1} R_{t-1}\,.$$

*Whether the shrinkage rate for $R$ in this equation is greater than or less than that in the equation without taxes is ambiguous. The shrinkage factor will increase with $\tau$ for $\tau$ near zero and $\alpha$ greater than .65 or so, but, naturally, as $\tau$ approaches 1, the incentive to concentrate consumption in the first period is so great that the shrinkage factor goes to zero. Equilibrium with this tax is not Pareto efficient, in the sense that a planner could make all generations better off than they are in this equilibrium. Agents perceive a tradeoff between current and future consumption that does not match the physical rate of transformation. Therefore a planner could offer an agent in this equilibrium higher $C_2$ in return for lower $C_1$, keep the agent's welfare constant, and end up with free resources to use benefitting another generation.*

*The second tax changes the budget constraints and adds a government budget constraint as follows:*

$$C_1(t) + P_t R_t + P_t(R_t - R_{t+1})\tau_t = (R_t - R_{t+1})^\alpha$$
$$C_2(t) = P_{t+1}R_{t+1} + T_{t+1}$$
$$T_t = \tau_t P_t(R_t - R_{t+1})\,.$$

*This leads to, as simplified FOC's,*

$$\frac{P_t}{P_{t+1}} = \frac{C_1(t)}{C_2(t+1)}$$
$$P_t = \frac{\alpha(R_t - R_{t+1})^{\alpha-1}}{1+\tau}\,.$$

*Note that we can already see here that the* $P_t/P_{t+1}$ *rate of transformation between current and future consumption perceived by the agent remains, in this equilibrium, equal to the physical rate of transformation, because* $\tau$ *affects current and future* $P$ *equiproportionately. The equilibrium is therefore, like the no-tax competitive equilibrium, Pareto efficient. The dynamics obtained when we solve the system for* $R$ *as before, become*

$$(1+\alpha)R_{t+1} - \frac{\alpha\tau}{1+\tau}R_{t+2} = R_t(1-\alpha)\,.$$

*This is a second order difference equation, with one unstable and one stable root (showing that one root must be unstable is a bit of a mess, but it is easy to check this for a couple of example values of* $\alpha$ *and* $\tau$*. The stable root is*

$$\frac{1-\alpha}{\frac{1+\alpha}{2}\left(1+\sqrt{1-4\frac{\alpha\tau(1-\alpha)}{(1+\tau)(1+\alpha)^2}}\right)}\,.$$

*It is not hard to see that this is necessarily larger than the shrinkage factor* $A = (1-\alpha)/(1+\alpha)$ *in the pure competitive solution, and thus results in slower exhaustion of the resource. The unstable root can't be present in the solution, as it would imply an infeasible path for* $R$.

(c) If the old as well as the young had access to the production technology, would the first generation simply use up all the resources? Why or why not?

*If the technology were linear and agents could produce* $C$ *from* $R$ *in both periods of life, the first generation would use up all of the resource. There would be no point in trading any of it to the young. But since the technology shows diminishing returns, and has the marginal product of* $R$ *going to infinity as the resource usage level goes to zero, agents who use all their* $R$ *endowment in production, leaving none for the young, would find the young willing to pay extremely high prices for* $R$*. It would thus be possible for this generation to increase its own consumption by withdrawing* $R$ *from their own production process and selling it to the young, for whom it has a higher marginal product. The equilibrium condition will be that* $R$ *has the same marginal product for all generations that are using it in production. If the old and young have the same production technology and are equal in population numbers, therefore, the old will sell more than half* *of their saved* $R$ *to the young (since the young and old will use up the same amount, and the young will save some).*