EXERCISE: CONSUMPTION SMOOTHING

We discussed the nature of a solution to the simple linearized international borrowing and lending model we considered in class, but didn't actually lay out a solution. In this problem, you will find the actual solution. The model has agent i, $i = 1, 2$ solving

$$\max_{C_i, B_i} E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_i(t)^{1-\gamma}}{1-\gamma}\right] \tag{1}$$

subject to

$$C_i(t) + B_i(t) = R_{t-1} B_i(t-1) + Y_i(t) \tag{2}$$

$$B_i(t) \geq -\bar{B}\,. \tag{3}$$

We assume that the bonds are privately issued, so that $B_1(t) = -B_2(t)$ is the market clearing condition. Assume that Y_1 and Y_2 are independent of each other and across time, with the same distribution and a mean $\bar{Y} > 0$.

(i) Linearize the model around a deterministic steady state in which $B = 0$, and solve for C_1, C_2, R, and B_1 as functions of the history of the exogenous processes Y_i.

(ii) Verify that in your linearized solution B and $C_1 - C_2$ are martingales and R is i.i.d.

The first order conditions for agent i are

$$\partial C: \qquad C_i(t)^{-\gamma} = \lambda_i(t)$$

$$\partial B: \qquad \lambda_i(t) = \beta R_t E_t \lambda_i(t+1)$$

$$TVC: \qquad \beta^t E[\lambda_i(t) B_i(t)] \xrightarrow[t\to\infty]{} 0\,.$$

Using the first of these FOC's to substitute out the λ_i's, Setting $B_t = B_1(t) = -B_2(t)$, and replacing the two individual budget constraints by their sum and their difference, we arrive at

$$C_1(t)^{-\gamma} = \beta R_t E_t[C_1(t+1)^{-\gamma}] \tag{A1}$$

$$C_2(t)^{-\gamma} = \beta R_t E_t[C_2(t+1)^{-\gamma}] \tag{A2}$$

$$\tilde{C}_t + B_t = R_{t-1} B_{t-1} + \tilde{Y}_t \tag{A3}$$

$$\hat{C}_t = \hat{Y}_t\,, \tag{A4}$$

where $\tilde{C}_t = (C_1(t) - C_2(t))/2$, $\hat{C}_t = (C_1(t) + C_2(t))/2$, and $\tilde{Y}_t$ and $\hat{Y}_t$ are defined analogously.

In the $B = 0$ deterministic steady state, $C_1(t) \equiv C_2(t) \equiv \bar{Y}$ and $R_t \equiv 1/\beta$. Linearizing about this steady state, replacing (A1) *and* (A2) *by their sum and difference, and*

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substituting out $\hat{C}$ *by using* (A4)*, we arrive at*

$$\frac{\gamma}{\bar{Y}} E_t d\tilde{C}_{t+1} = \frac{\gamma}{\bar{Y}} d\tilde{C}_t \tag{A5}$$

$$\frac{\gamma}{\bar{Y}} E_t d\hat{Y}_{t+1} = \frac{\gamma}{\bar{Y}} d\hat{Y}_t + \beta dR_t \tag{A6}$$

$$d\tilde{C}_t + dB_t = \beta^{-1} dB_{t-1} + \tilde{Y}_t \, . \tag{A7}$$

Notice that (A5) *has already delivered the conclusion that* $\tilde{C}$ *is a martingale. Because of the i.i.d. assumptions on the* Y_i*'s,* (A6) *can be solved for* dR_t *as*

$$dR_t = \frac{\gamma}{\beta \bar{Y}} (\bar{Y} - Y_t) \, .$$

Since the only forward-looking part of the system is now (A7)*, we can solve the model without matrix methods. Solving forward and using the martingale property for* $d\tilde{C}$*, we get*

$$dB_t = \frac{\beta}{1-\beta} d\tilde{C}_t$$

or equivalently

$$d\tilde{C}_t = (\bar{R} - 1) dB_t \, . \tag{A8}$$

This shows that B *also will have the martingale property.*

However we still have not shown how the Y*'s map into the* C*'s. Substituting our solutions for* R_t *in terms of* $\hat{Y}$ *and for* B *in terms of* $\tilde{C}$ *into* (A5)*, we arrive at*

$$dB_t = dB_{t-1} + \beta d\tilde{Y}_t \, .$$

If we start from $t = 0$*, then we need an initial value for* dB_{-1} *to determine the solution. Given that, this equation determines the time path of* B *from the* Y*'s, then we find* $\tilde{C}$ *from* (A8)*. Of course* $\hat{C}$ *has already been determined by* (A4)*.*