Econ 504, part II
Spring 2001
Linearization; Barro Model and Extensions*

Linearization with nonstationarity and unbounded disturbances

Suppose we have a dynamic system

$$x_{t+1} = f(x_t, \varepsilon_{t+1}) \,. \tag{1}$$

We can apply it recursively to obtain

$$x_{t+1} = f(f(x_{t-1}, \varepsilon_t), \varepsilon_{t+1}) \,. \tag{2}$$

Assuming the usual conditions are met, we can take a Taylor expansion of it in $\varepsilon_t, \varepsilon_{t+1}$ in the neighborhood of $\varepsilon_t = \varepsilon_{t+1} = 0$ and $x_{t-1} = x^*$ by applying the chain rule for differentiating functions of functions:

$$\begin{aligned} x_{t+1} - f(f(x^*, 0), 0) \doteq \\ D_1 f(f(x^*, 0), 0) D_2 f(x^*, 0)\varepsilon_t + D_2 f(f(x^*), 0)\varepsilon_{t+1} \\ + D_1 f(f(x^*, 0), 0) D_1 f(x^*, 0)(x_{t-1} - x^*) \,. \end{aligned} \tag{3}$$

This expansion will be, like any Taylor expansion, accurate for small enough ε_t and ε_{t+1}. The error of approximation will grow small relative to the size of the ε's as the ε's get small.

If $x^* = \bar{x}$, where $\bar{x} = f(\bar{x}, 0)$ (i.e. $\bar{x}$ is a deterministic steady state value for x_t), We can apply these ideas recursively to get the s-step Taylor expansion

$$x_t - \bar{x} = \sum_{s=0}^{t-1} D_1 f(\bar{x}, 0)^s D_2 f(\bar{x}, 0)\varepsilon_{t-s} + D_1 f(\bar{x}, 0)^t (x_0 - \bar{x}) \,. \tag{4}$$

- This formula does require $x^* = \bar{x}$. Otherwise, we would obtain a much more complicated expression involving derivatives at many different values of x. Such a more complicated "linearization about a deterministic path" can sometimes be useful, but it is not what has been done in most applications of linearization to macroeconomic equilibrium models.
- The validity of the Taylor expansion for small ε and small deviations of x_0 from $\bar{x}$ does not depend on any condition on the eigenvalues of $D_1 f(\bar{x}, 0)$.
- If we want to take a limit as $t \to \infty$ and still get a valid approximation, we do require that all the eigenvalues of $D_1 f(\bar{x}, 0)$ be less than one in absolute value, as otherwise the limiting expression would not be defined. If this condition is met, and if we can uniformly bound all the ε_t's, we can make Taylor-series-approximation accuracy claims as the bound on the size of the ε's shrinks in this $t \to \infty$ case.
- If the ε's are random and have unbounded support, e.g. i.i.d. $N(0, \sigma^2)$, we will not be able to make non-probabilistic accuracy claims. There will always be some probability that ε's are large enough to invalidate the Taylor approximation, no matter how small we make σ.
- In this case we also cannot let $t \to \infty$ and make accuracy claims. The probability that *eventually* some ε will be large enough to invalidate the approximation is one, no matter how small is λ. Once we are out of the range where the approximation is valid, the true nonlinear system might be explosive or tend toward another steady state, so the approximation would be come permanently bad at that time.
- So with unbounded stochastic disturbances, our claims about the approximation take the form, "If σ is small enough, the probability that the linearized model is accurate over the time span $t = 0, \ldots, T$ is greater than $1 - \delta$." And by making σ small enough, we can make T as large and δ as small as we like.
- This kind of approximation claim can be made regardless of the eigenvalues of $D1f$.
- Nonetheless, the eigenvalues matter. If they are all small in absolute value, the T we can obtain with a given σ and δ will grow rapidly as $\sigma \to 0$. If the largest is much bigger than one, the T will grow extremely slowly as $\sigma \to 0$, so that the approximation in practice is only good over very short time spans. Roots near one in absolute value are intermediate cases. They will allow T to grow as $\sigma \to 0$ much faster than for roots much greater than one, but much slower than for roots much less than one.

Two examples

Here are two examples of simulated time paths of nonlinear models and their linearizations. In one, the linearization is stable and the nonlinear model globally unstable, while in the other the linearization is unstable but the nonlinear model is globally stable. In both the linearization is quite accurate for a time (note the very different amounts of time), but then deteriorates catastrophically.

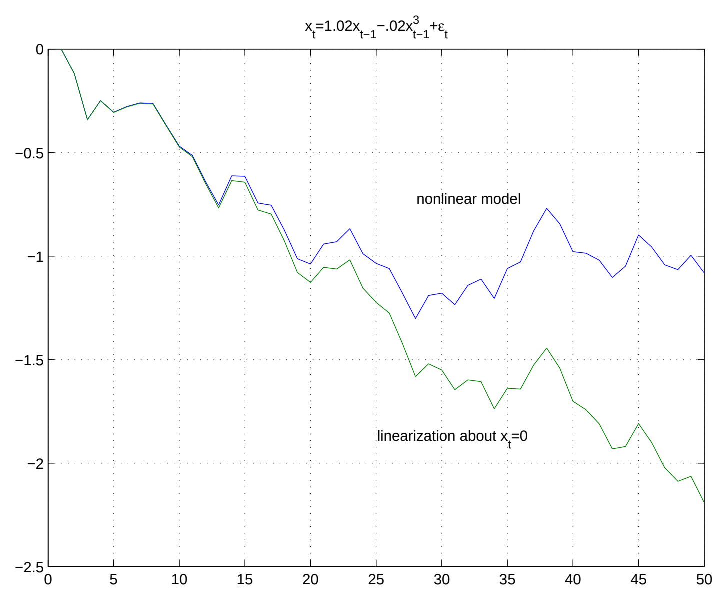

Figure 1:

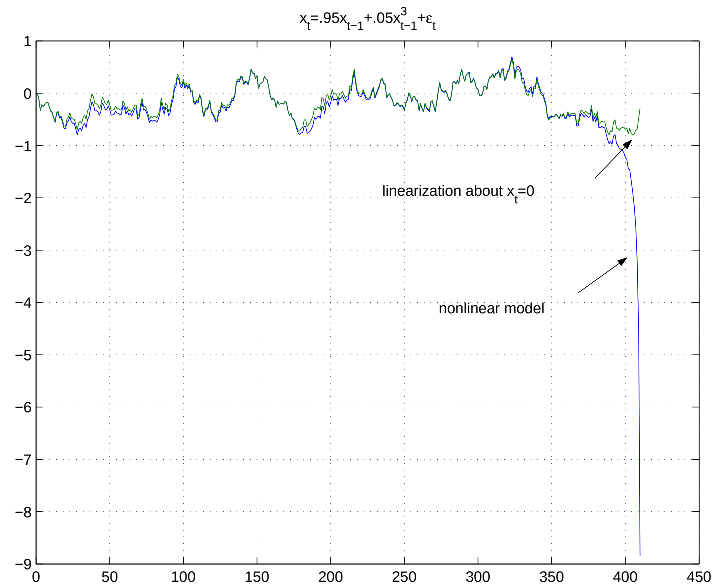

Figure 2:

The Barro model

This section generalizes the discussion in lecture (and in Barro's paper) to consider non-quadratic deadweight loss.

The Barro tax-smoothing model concludes that it is optimal not to plan to pay down existing public debt, so that both taxes and debt are expected to stay constant under optimal policy. It employs drastic simplifying assumptions, making its mathematics simple, indeed equivalent to the structure of the permanent income model. Note that in assuming that the deadweight loss from constant revenues is constant, this model is dealing with something like a labor tax, not a capital tax.

Government's problem

$$\text{Obj.fcn:} \qquad \max_{\tau_s, B_s} E\left[\sum_{t=0}^{\infty} \ell(\tau_t)\beta^t\right] \tag{5}$$

s.t.

$$\lambda: \qquad B_t + \tau_t \geq \rho_{t-1} B_{t-1} + G_t \tag{6}$$

$$\text{Fisher:} \qquad \rho_t = \beta^{-1} \tag{7}$$

$$\text{no Ponzi:} \qquad E[\beta^t B_t] \underset{t \to \infty}{\to} = 0 \tag{8}$$

Note that (7) implies absence of risk aversion, or else that consumption is constant. By using a function of τ alone to stand for deadweight loss from taxes and τ to stand for total revenues, the model avoids treating explicitly fluctuations in the tax base and possible effects of tax rates on the tax base.

FOC's

$$\partial\tau: \qquad \ell'(\tau_t) = \lambda_t$$

$$\partial B: \qquad \lambda_t = E_t \lambda_{t+1}$$

Discussion The FOC's deliver immediately the conclusion that $\ell'(\tau_t)$ is a martingale. If ℓ' is everywhere positive, this implies that in the absence of uncertainty, τ itself will be set at a constant. The government's no-Ponzi constraint (8) together with its budget constraint require that the planned constant level of taxes match the interest rate times current debt plus the discounted present value of future G. That is,

$$\tau_t = (\beta^{-1} - 1)\left(B_t + \left[\sum_{s=0}^{\infty} \beta^s G_{t+s}\right]\right) \tag{9}$$

With uncertainty, however, the difference between martingale behavior for τ itself and for $\ell'(\tau)$ can be important. Of course if ℓ is quadratic, then ℓ' is linear and τ itself is implied to be a martingale. But it is natural to suppose that ℓ must, at least for high values of τ, increase more rapidly than a quadratic. This is because tax rates and current output are inherently bounded, and one supposes that as tax revenues approach their maximum, the deadweight loss must increase very rapidly. So we might assume that ℓ' is a convex function of τ, and thus that τ is a concave function of ℓ'. Because of the fact that if $Ex = \bar{x}$, $Ef(x) \leq f(\bar{x})$ for any concave function f, the martingale property for ℓ' implies, under convexity of ℓ', $E\tau_{t+1} \leq \tau_t$. That is, in the presence of uncertainty expected revenues decline over time. In effect, uncertainty makes it optimal to tax more heavily now as insurance against being driven to very inefficient high tax rates in the future.

Even in the quadratic case considered by Barro, though expected future τ is constant, actual τ adjusts, period-by-period, to any random disturbances in current and future G. It is not hard to verify that under these conditions B, like τ, is a martingale.

Adding a price level to the Barro model

Model Same objective function (5) as before. Constraints now

$$\lambda: \qquad \frac{B_t}{P_t} + \tau_t \geq \rho_{t-1}\frac{B_{t-1}}{P_t} + G_t \tag{10}$$

$$\mu: \qquad 1 \leq \beta\rho_t E_t\left[\frac{P_t}{P_{t+1}}\right] \tag{11}$$

$$\text{no Ponzi:} \qquad \lim_{t\to\infty} E\left[\beta^t \frac{B_t}{P_t}\right] = 0 \text{ or } B_t \geq 0 \tag{12}$$

FOC's

$$\partial\tau: \qquad \ell'(t) = \lambda_t$$

$$\partial B: \qquad \lambda_t = \beta\rho_t E_t\left[\lambda_{t+1}\frac{P_t}{P_{t+1}}\right]$$

$$\partial\rho: \qquad \beta B_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] = \mu_t\beta E_t\left[\frac{P_t}{P_{t+1}}\right]$$

$$\partial P: \qquad \frac{\lambda_t}{P_t^2}\left(B_t - \rho_{t-1}B_{t-1}\right)$$
$$= \mu_t\rho_t\beta E_t\left[\frac{1}{P_{t+1}}\right] - \mu_{t-1}\rho_{t-1}\frac{P_{t-1}}{P_t^2}$$

Algebra From the B and ρ FOC's, together with the private FOC (11), we can derive a relation of μ_t to λ_t:

$$\mu_t = \frac{B_t}{P_t}\lambda_t \tag{13}$$

Using this and (11) again in the P FOC, we get

$$\lambda_t(B_t - \rho_{t-1}B_{t-1}) = \lambda_t B_t - \lambda_{t-1}B_{t-1}\rho_{t-1}\,,$$

which reduces to $\lambda_t = \lambda_{t-1}$, and thus via the τ FOC to $\tau_t = \tau_{t-1}$.

Discussion The conclusion we have derived here applies only *after* $t = 0$. This is because it uses, in time-t FOC's, expectational constraints dated $t-1$. In particular, the FOC w.r.t. P_0 is not the P FOC given above with t set to zero, but instead

$$\partial P_0: \qquad \frac{\lambda_0}{P_0^2}\left(B_0 - \rho_{-1}B_{-1}\right) = \mu_0\rho_0\beta E_0\left[\frac{1}{P_1}\right].$$

which reduces to $\lambda_0 \rho_{-1} B_{-1}/P_0^2 = 0$. We postpone momentarily discussing the implications of this initial-date condition. Also, we have assumed an interior solution. If there is a $B \geq 0$ no-Ponzi condition, and we were up against this constraint last period, the $\tau_t = \tau_{t-1}$ conclusion no longer applies. We postpone discussion of this temporarily also.

The intertemporal budget constraint of the government, which applies because of the no-Ponzi condition, requires that τ_0 be set to satisfy the same condition (9) as before, now interpreted as holding in expectation instead of exactly, and with B/P replacing B. How then does the government maintain fiscal balance, with τ_t constant but future G's changing stochastically? It need "do" nothing. The price level will adjust automatically to maintain the fiscal intertemporal budget constraint. A surprise increase in expected future G's leads to a surprise increase in P just sufficient to reduce B/P be enough to maintain intertemporal budget balance.

The first period At $t = 0$, optimal policy satisfies the P_0 FOC above. For this to hold, with B_{t-1} and ρ_{t-1} non-zero, requires $\lambda_0 = 0$ or $P_0 = \infty$. $\lambda_t = 0$ is possible only at $\tau = 0$, where the marginal deadweight loss from taxation is zero. That in turn is possible only if debt is zero and current G is zero. If debt is issued at $t = 0$, then at $t = 1$ we will have $\tau_1 = \tau_0 = 0$, and this condition persists as long as $B_t > 0$. But with zero taxes and $G > 0$, there is no way that B can decrease, so this policy is unsustainable. If $\tau_0 > 0$, then the P_0 FOC requires $P_0 = \infty$, which is not technically possible. If P is ever infinite, then the future rates of inflation that enter into the model's equation are all undefined. However, it is true that welfare is higher the higher the initial P, so that optimal policy under the more realistic $B \geq 0$ constraint is in fact a very large initial surprise inflation that all but wipes out the real value of B_{-1}. Under the limit form of the no-Ponzi constraint, optimal policy is to switch the sign of B, converting outstanding government debt to government assets, in which holders make payments to the government instead of vice versa. The optimal initial P is then chosen so as to make earnings from the new government assets sufficient to cover the discounted present value of future G.

Suppose $B_t = 0$. It can be optimal to cover G_t plus any outstanding debt entirely with τ_t, leaving $B_t = 0$. However, it can be shown that this cannot be optimal if the expected ℓ'_{t+1} is below the current ℓ'_t. That is, it can be optimal not to borrow only if we can do so without pushing τ up to "above-normal" levels. In particular, if G is unusually high, it will always be optimal to borrow. From then on, it will be optimal to keep τ constant, unless it is not feasible to do so. In this latter case it will be optimal to inflate away essentially all the real value of the debt again, then set τ_t to a new, higher level, that will again be maintained constant until the next dire fiscal emergency that cannot be covered with an unexpected inflation tax on debt-holders. More detailed discussion of these "debt repudiation dynamics" are in my paper "Fiscal consequences for Mexico of adopting the dollar", on my website and on the course reading list.

Remarks about realism Barro's model and its extension to consider inflation are both drastic simplifications, meant to illustrate a mechanism that should be taken seriously. Taxes should generally be smoothed, to some extent. Unanticipated inflation (and, in a model with long-term debt, unanticipated interest rate changes) are available as tools to smooth taxes. In reality unanticipated inflation has costs that are not incorporated in these models, because of price stickiness, money illusion, contracts written without inflation contingencies, etc. These have to be balanced off against the benefits of using unanticipated inflation to smooth taxes.