Final Exam Answers, problems 2-5

(2) (45 minutes) Consider a model in which private agents face a tax on gross investment. The representative agent maximizes

$$E\left[\sum_{t=0}^{\infty}\beta^t \log C_t\right] \tag{11}$$

subject to

$$C_t + B_t + (1+\tau_t)(K_t - \delta K_{t-1}) = A_t K_{t-1}^{\alpha} + g_t + R_{t-1}B_{t-1}\,. \tag{12}$$

Here $0 < \delta < 1$ and $0 < \alpha < 1$. The private agent also faces the no-Ponzi condition $B_t \geq 0$, all t. The government faces a budget constraint

$$B_t + \tau_t(K_t - \delta K_{t-1}) = R_{t-1}B_{t-1} + g_t\,. \tag{13}$$

It takes the private sector's competitive optimizing behavior (including its transversality conditions) as a constraint on its actions, and it treats the stream of transfer payments g_t as exogenously given. In your discussion you can assume that there is an equilibrium with $B_t > 0$ for all t. The government's objective is to maximize private welfare as defined by (11) by choosing the tax rates.

(a) Display the first order conditions for optimizing private sector behavior.

$$\begin{aligned}
\partial C: &\quad \frac{1}{C_t} = \lambda_t \\
\partial B: &\quad \lambda_t = \beta R_t E_t[\lambda_{t+1}] \\
\partial K: &\quad (1+\tau_t)\lambda_t = \beta E_t\left[\lambda_{t+1}\big((1+\tau_{t+1})\delta + \alpha A_{t+1}K_t^{\alpha-1}\big)\right]\,.
\end{aligned}$$

We can easily solve to eliminate λ:

$$\begin{aligned}
\psi_B: &\quad \frac{1}{C_t} = \beta R_t E_t\left[\frac{1}{C_{t+1}}\right] \\
\psi_K: &\quad \frac{1+\tau_t}{C_t} = \beta E_t\left[\frac{(1+\tau_{t+1})\delta + \alpha A_{t+1}K_t^{\alpha-1}}{C_{t+1}}\right]\,.
\end{aligned}$$

TVC's, which are part of the first order conditions, are

$$E\left[\beta^t \frac{(1+\tau_t)K_t}{C_t}\right] \xrightarrow[t\to\infty]{} 0, \qquad E\left[\beta^t \frac{B_t}{C_t}\right] \xrightarrow[t\to\infty]{} 0\,.$$

(b) Display the first order conditions for the optimizing government, assuming that the government can commit to future policies and will be believed when it announces its commitments.

We simplify things somewhat by using the government budget constraint in the private constraint to obtain the social resource constraint

$$\mu: \qquad C_t + K_t - \delta K_{t-1} = A_{t+1} K_t^{\alpha} \,.$$

We will use μ as the government's Lagrange multiplier for the SRC and ν for (13). The LM's for the two private FOC's are the ψ_B and ψ_K displayed next to them. The government's FOC's then are, for periods $t > 0$,

$$\partial C: \qquad \frac{1}{C_t} = \mu_t - \frac{1}{C_t^2}\Big(\psi_B(t) - R_{t-1}\psi_B(t-1) + \psi_K(t)(1+\tau_t) - \psi_K(t-1)\big((1+\tau_t)\delta + \alpha A_t K_{t-1}^{\alpha-1}\big)\Big)$$

$$\partial K: \qquad \mu_t + \tau_t \nu_t = \beta E_t \left[\mu_{t+1}\big(\alpha A_{t+1} K_t^{\alpha-1} + \delta\big) + \delta \tau_{t+1}\nu_{t+1} + \psi_K(t)\alpha(\alpha-1)\frac{A_{t+1}K_t^{\alpha-2}}{C_{t+1}}\right]$$

$$\partial B: \qquad \nu_t = \beta R_t E_t \nu_{t+1}$$

$$\partial R: \qquad \beta E_t[B_t \nu_{t+1} + \frac{\psi_B(t)}{C_{t+1}}] = 0$$

$$\partial \tau: \qquad \nu_t(K_t - \delta K_{t-1}) = \frac{1}{C_t}(\psi_K(t) - \delta\psi_K(t-1)) \,.$$

(c) Does the government face a time-consistency problem here? Use the first order conditions of the government's problem to explain your answer. Would your answer change in the case of $\delta = 0$ (100% depreciation)?

The FOC's at $t = 0$ differ from those at $t > 0$ because at the initial date the government is not bound to behave in a way conforming to expectations that were formed at $t = -1$. This means that all terms in the FOC's that are multiplied by Lagrange multipliers dated $t-1$ disappear for $t = 0$. The ∂C and $\partial\tau$ FOC's contain LM's dated $t-1$, so there is a time-consistency problem. A time-consistency problem is just a name for the situation where behavior differs, given the same state of the economy, according to what the absolute date t is. With $\delta = 0$, the $t-1$-dated LM would not appear in the $\partial\tau$ FOC, but there would still be such a term in the ∂C FOC, so the time-consistency problem would still be there. A sophisticated answer to this question would have raised the question of whether ψ_B and ψ_K might possibly be identically zero in equilibrium, since if that were true there would not actually be any time-inconsistency. But it is not hard to show that if $\psi_B \equiv \psi_K \equiv 0$, then also $\nu \equiv 0$. This in turn makes $\mu_t = 1/C_t$, so that the government's ∂K FOC becomes the FOC of the model without any distorting tax. But the private ∂K FOC would then directly contradict the government's. So these LM's are not identically zero in equilibrium.

(d) In class we considered a model in which an optimizing government that can commit wishes to make capital taxes high immediately, while promising low future capital taxes. Is it similarly optimal here to set the tax rate high initially while promising low future taxes? Explain your answer.

This question is difficult to tackle formally under a time constraint. A good informal answer would note that a pure capital tax, imposed as a surprise at $t = 0$ and accompanied by credible commitments not to do capital taxation in the future, does not distort current investment decisions. Current investment decisions relate the current cost of capital goods to the discounted present value of future returns, and therefore are unaffected by changes to current returns to past investments, unless those changes have implications for expected future returns. The investment tax, though, directly affects the current price of capital goods. An investment tax imposed now, accompanied by a promise not to tax in the future, does, therefore, distort current investment decisions. The distortion is in some sense worse if the tax is temporary, assuming $\delta > 0$. Investors contemplating long term investments and confronting a single-period investment tax will be likely to postpone their investments by one period to avoid the tax. If the tax is permanent, the incentive to postpone will not be there. In fact, if a current tax is accompanied by a promise to impose *higher* taxes in the future, it is possible to eliminate any negative effect of the tax on current investment. Of course such a policy implies, eventually, higher distortion, because the tax rate cannot go on rising forever and must eventually stabilize at a rate high enough to cover spending plus debt service.

(3) (45 minutes) Consider a "two-country" model in which each country's government issues nominal bonds denominated in its own monetary unit, backed by a proportional tax on endowment income, and these bonds are internationally traded. That is, the budget constraints of representative agents in countries 1 and 2 are

$$C_1(t)+\frac{B_{11}(t)}{P_1(t)}+\frac{B_{12}(t)}{P_2(t)} = (1-\tau)Y_1(t)+g+\frac{RB_{11}(t-1)}{P_1(t)}+\frac{RB_{12}(t-1)}{P_2(t)} \tag{14}$$

$$C_2(t)+\frac{B_{21}(t)}{P_1(t)}+\frac{B_{22}(t)}{P_2(t)} = (1-\tau)Y_2(t)+g+\frac{RB_{21}(t-1)}{P_1(t)}+\frac{RB_{22}(t-1)}{P_2(t)}\,. \tag{15}$$

The representative agent in each country i maximizes the expected discounted sum of $\log C_i(t)$, where discounting is at the rate β, with $R\beta = 1$. Assume that $Y_1(t)$ and $Y_2(t)$ are i.i.d. across time and independent of each other.

The government budget constraints are

$$B_{11}(t) + B_{21}(t) + P_1(t)\tau Y_1(t) = P_1(t)g + R\big(B_{11}(t-1) + B_{21}(t-1)\big) \tag{16}$$

$$B_{12}(t) + B_{22}(t) + P_2(t)\tau Y_2(t) = P_2(t)g + R\big(B_{12}(t-1) + B_{22}(t-1)\big) \tag{17}$$

We assume that each government takes it as a constraint that it must issue only non-negative amounts of debt and cannot acquire debt of the other country. Individuals, however, can if necessary hold negative amounts of government debt — essentially doing international lending on terms that mimic those of government debt. Note that the notation implicitly sets policy to keep the tax rate, the nominal interest rate and the level of real expenditures constant in both countries.

This question should have specified some sort of no-Ponzi condition for each private agent. The TVC's of the other-country agent of course rule out excessive borrowing as equilibria, but unless individuals see some borrowing constraint, they will not be satisfied with any bounded consumption path as a competitive equilibrium. However, no one seem to notice this in answering the question. Any sort of no-Ponzi condition that ended up not binding in equilibrium would suffice.

(a) Find the first order conditions for an optimum for the private agents.

For agent 1 the conditions are

$$\partial C: \qquad \frac{1}{C_1(t)} = \lambda_1(t)$$

$$\partial B_{11}: \qquad \frac{\lambda_1(t)}{P_1(t)} = \beta R E_t\left[\frac{\lambda_1(t+1)}{P_1(t+1)}\right]$$

$$\partial B_{12}: \qquad \frac{\lambda_1(t)}{P_2(t)} = \beta R E_t\left[\frac{\lambda_1(t+1)}{P_2(t+1)}\right]$$

The TVC's, which are not necessary for the rest of the problem, and which only a few people even tried to make explicit, are tricky here, because $B_{ij}(t) \geq 0$ is *not* imposed on the solution, so the usual substitution of $-B$ for dB is not automatically possible here. If we impose the simple form of borrowing constraint $B_{11}(t)/P_1 + B_{12}(t)/P_2 \geq 0$, then the TVC w.r.t. B_{11}, for example, becomes

$$\limsup_{t\to\infty} E\left[-\beta^t \frac{-\lambda_1(t)dB_{11}(t)}{P_1(t)}\right] \leq 0\,,$$

for which a sufficient condition is

$$\lim_{t\to\infty} E\left[\beta^t \frac{1}{C_1(t)}\left(\frac{B_{11}(t)}{P_1(t)} + \frac{B_{12}(t)}{P_2(t)}\right)\right] = 0\,.$$

(b) Find the optimal allocation that would be produced by a planner who put equal weight on the welfare of both countries' agents and whose constraint was only the social resource constraint implied by the model's constraints.

Adding the two private budget constraints, then substituting in the two government budget constraints, gives us the same social resource constraint we studied in classroom and exercise models:

$$\zeta: \qquad C_1(t) + C_2(t) = Y_1(t) + Y_2(t) \,.$$

It is then easy to verify, using exactly the methods shown in lectures or exercise answers, that the planner will set $C_1(t) = C_2(t) = (Y_1(t) + Y_2(t))/2$ at every date t.

(c) Determine whether with only these two kinds of nominal debt traded, and the constant τ, R and g policy, the planner's allocation can emerge as a competitive equilibrium. Justify your answer.

This question is hard, but has a well-defined answer that conceivably could have been arrived at in the time allotted.

The conceptual framework for the answer can be explained informally. As in our classroom examples with bonds-only and with two traded equities whose dividends are Y_i, $i = 1, 2$, we need to verify that the $C_1 \equiv C_2$ equilibrium can be implemented as a competitive equilibrium without making relative wealths explode. As in those models, it will turn out that relative wealths do explode under $C_1 \equiv C_2$ unless portfolios are arranged so that the returns on assets held just offset differences in endowment streams. As we discussed in class, in discrete time we generally will need infinitely many assets to arrange this, unless the assets can be defined to have just the right pattern of returns. We saw there that with log utility (but not more general CRRA utility), the two "equity-in-endowment-stream" assets were adequate to implement the planner's equilibrium, because the planner's equilibrium involves $C_1 \equiv C_2$ and thus requires the asset yields to offset exactly the income differential $Y_1 - Y_2$, which they will if the assets are held in the proper amounts.

In the model at hand, we have two assets as in the case of the two-equity model, and the problem then is to verify that their returns can in fact exactly offset $Y_1 - Y_2$. Denoting the total outstanding government debt of country i at time t as $B_i(t) = B_{1i}(t) + B_{2i}(t)$, and the real value of country i debt at time t as $b_i(t) = B_i(t)/P_i(t)$, We can rearrange the country-i government budget constraint, divide it by $P_i(t)C_i(t)$, and apply the E_{t-1} operator to obtain

$$\frac{b_i(t-1)}{C_i(t-1)} R E_{t-1}\left[\frac{C_i(t-1)P_i(t-1)}{C_i(t)P_i((t)}\right] = R\frac{b_i(t-1)}{C_i(t-1)} = E_{t-1}\left[\frac{b_i(t)}{C_i(t)} + \frac{\tau Y_i(t) - g}{C_i(t)}\right] . \qquad (**)$$

Because the sum of the two private net asset positions is $b_1 + b_2$ at every date, and because neither b_1 nor b_2 can be negative, private transversality will impose that $E[\beta^t b_j(t)/C(t)] \to 0$, where we have dropped the subscript on C because we are examining the planner's allocation with $C_1 \equiv C_2$. This means that we can solve $(**)$ forward to obtain

$$\frac{b_i(t)}{C(t)} = E\left[\frac{\tau Y_i(t) - g}{C(t)}\right] .$$

Since in our candidate allocation $C(t) = .5 \cdot \big(Y_1(t) + Y_2(t)\big)$, and since $Y_i(t)$ is i.i.d. both over t and over i, the right-hand-side of the expression above is a constant, so the left-hand side, $b_i(t)/C(t)$, is also a constant, which we will call Θ. With this constant in hand, we can now rewrite the original budget constraint, divided through by $C(t)$ as

$$\Theta = R\Theta\frac{C(t-1)P_i(t-1)}{C(t)P_i(t)} + \frac{g - \tau Y_i(t)}{C(t)} .$$

Multiplying this through by $C(t)/(\Theta C(t{-}1)$ and rearranging, we can obtain an expression for the return on government debt of country i:

$$R\frac{P_i(t-1)}{P_i(t)} = \frac{C(t)}{C(t-1)} - \frac{g - \tau Y_i(t)}{\Theta C(t-1)} .$$

The yield differential between the two types of government debt is then

$$\frac{2\tau}{\Theta} \cdot \frac{Y_1(t) - Y_2(t)}{Y_1(t-1) + Y_2(t-1)} . \tag{$\dagger$}$$

Now we can see that the yield differential on the two types of government debt is linear in $Y_1(t) - Y_2(t)$ and depends on nothing else dated t. Thus it will be possible, if the two agents begin with the same net worth, for them to trade bonds with each other until they reach portfolios that make the difference in their portfolio yields exactly offset the difference between their endowment income and $C(t)$. They will therefore each period use the randomness in their portfolio yields to maintain consumption equality, and their net worths will remain identical. Unlike the case of the two-equities model, the agents here will have to do some asset trading to rebalance portfolios each period to achieve the perfect hedging the equilibrium requires. This is required because the governments will be issuing or retiring debt, because the differential movements in prices will produce shifts in real portfolio compositions, and because the lagged Y terms in $(\dagger)$ in themselves create a need for rescaling of positions.

To demonstrate formally the need for exact matching of asset returns to endowment differentials, the argument runs parallel to the versions of the model discussed in class. Taking the difference of the two private budget constraints (and using $C_1(t) = C_2(t) = C(t)$) gives us

$$
\begin{aligned}
C_1(t) - C_2(t) = 0 \\
&= \frac{B_{21}(t) - B_{11}(t)}{P_1(t)} - \frac{B_{12}(t) - B_{22}(t)}{P_2(t)} \\
&\quad - R\left(\frac{B_{21}(t-1) - B_{11}(t-1)}{P_1(t)} - \frac{B_{12}(t-1) - B_{22}(t-1)}{P_2(t)}\right) + Y_1(t) - Y_2(t)\,.
\end{aligned}
$$

Dividing through by $C(t)$ and applying E_{t-1} gives us

$$
\begin{aligned}
R\bigg(\frac{B_{21}(t-1) - B_{11}(t-1)}{P_1(t-1)} - \frac{B_{12}(t-1) - B_{22}(t-1)}{P_2(t-1)}\bigg) \\
= E_{t-1}\left[\frac{B_{21}(t) - B_{11}(t)}{P_1(t)} - \frac{B_{12}(t) - B_{22}(t)}{P_2(t)}\right]\,,
\end{aligned}
$$

where we have used the fact that with Y_1 and Y_2 independent of each other and identically distributed, $E[Y_1(t) - Y_2(t)/C(t)] = 0$. So we have arrived at an unstable difference equation, which will violate private transversality unless

$$
\frac{B_{21}(t)}{P_1(t)} + \frac{B_{22}(t)}{P_2(t)} \equiv \frac{B_{12}(t)}{P_2(t)} + \frac{B_{11}(t)}{P_1(t)}\,.
$$

That is, the real value of the holdings of newly purchased foreign and domestic debt must be the same in the two countries at all dates.

(4) (20 minutes) Answer one of the following 3 questions:

These questions were all matters of making intelligent remarks about course readings, and do not require elaboration here.

(5) (25 minutes) An island economy with one farmer can invest in clearing land to make it productive, but it has only a finite amount of it, and it faces increasing costs of land clearing. Letting K represent cleared land, its technology is

$$
C_t + \frac{K_t - K_{t-1}}{(1-K_t)(1-K_{t-1})} = A_t K_{t-1}^{\alpha}\,. \tag{18}
$$

It is impossible to "unclear" land, so an additional constraint is $K_t \geq K_{t-1}$, all t. The farmer's objective is to maximize the discounted sum of expected log of consumption C, discounting at the rate β.

(a) Does the farmer's problem satisfy conditions that imply that a solution to his Euler equations and transversality condition must be an optimum?

(b) What is (or are) the Euler equations for this problem?

(c) What is the transversality condition here? Obviously it does not have the usual interpretation as ruling out excessive accumulation of K (since that is ruled out by the technology). What is its behavioral interpretation; that is, what kind of suboptimal behavior does it rule out?

(a) The constraint is naturally construed as an inequality, and can be written as

$$C_t + \frac{1}{1-K_t} - \frac{1}{1-K_{t-1}} - A_t K_{t-1}^{\alpha} \le 0\,.$$

At each t the constraint is not convex in the choice variables K_t, K_{t-1} and C_t for all possible values of those variables as can be checked by noting that the second derivative matrix of the constraint with respect to its three choice-variable arguments is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{2}{(1-K_t)^3} & 0 \\ 0 & 0 & -\dfrac{2}{(1-K_{t-1})^3} + \alpha(1-\alpha)K_{t-1}^{\alpha-2} \end{bmatrix}.$$

The term in the lower right corner is negative for K_{t-1} close to 1, so the constraint function is not globally convex. It is also not quasi-convex. Thus, even though log consumption is concave in C, and the objective function is therefore concave, it may appear that checking Euler equations and transversality conditions is only possibly, not certainly, going to give us an optimum. Stating this was an OK answer.

However, there is a bit of cleverness available that it was unreasonable to expect on a 3-hour exam, and that I didn't realize myself was needed until I worked more carefully on the problem in preparation for grading it. If we define $X_t = 1/(1-K_t)$, the constraint can be written

$$C_t + X_t - X_{t-1} - A_t\left(1 - \frac{1}{X_{t-1}}\right)^{\alpha} \le 0\,.$$

It turns out that in this form the constraint is after all convex in C_t, X_t, and X_{t-1} jointly. It is linear in C_t and X_t, and it is easy to check that the second derivative with respect to X_{t-1} is positive, making the constraint convex it its three arguments. Transforming variables can change the concavity and convexity properties of objective functions and constraints, but it does not change the solutions to Euler equations or transversality conditions. Thus the fact is that the solution to the Euler equations and the transversality condition are necessarily the unique optimal solution.

(b) The Euler equations are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t$$

$$\partial K: \qquad \frac{\lambda_t}{(1-K_t)^2} - \mu_t = \beta E_t\left[\lambda_{t+1}\left(\frac{1}{(1-K_t)^2} + \alpha A_{t+1}K_t^{\alpha-1}\right) - \mu_{t+1}\right]$$

The negative signs on μ_t, the LM on $K_{t-1} - K_t \le 0$, arise from the need to make it a less-than inequality if we want its LM to be non-negative.

(c) The transversality condition is

$$\limsup_{t\to\infty}\left\{-\beta^t E\left[\left(\frac{\lambda_t}{(1-K_t)}-\mu_t\right)(K_t^*-K_t)\right]\right\}\leq 0\,.$$

Because the technology imposes $K > 0$, and $\lambda_t > 0$ also, this can be simplified, for the case where $\mu_t = 0$ for all $t > T$ for some date T, to

$$\lim_{t\to\infty}\beta^t E\left[\left(\frac{\lambda_t}{1-K_t}\right)K_t\right]=0\,.$$

Since $\lambda_t = 1/C_t$, this TVC is still ruling out overaccumulation, but it rules out maintenance of high and growing values of $K/(1-K)$ relative to C, instead of ruling out high and growing values of K/C as in standard growth models. This is natural, because the technology implies that additional units of cleared land are increasingly expensive, in terms of consumption goods, as K approaches 1. The μ_t component of the TVC tells us that in a sequence of periods when the $K_t \geq K_{t-1}$ constraint is binding, $K/\big((1-K)C\big)$ can grow faster. If such periods are intermittent, they have little effect on the TVC in practice, since it is a $\limsup$ condition. If a steadily declining A leads to $\mu_t > 0$ for all $t > T$, the TVC is importantly affected. It is not necessarily suboptimal for $K/\big((1-K)C\big)$ to grow rapidly if this is occurring with K constant and C steadily declining.