Exercise: Linearizing an RBC*

Consider a model in which the representative consumer solves

$$\max_{\{C_s,L_s,B_s\}} E\left[\sum_{t=0}^{\infty}\beta^t\frac{\left(C_t^{\theta}(1-L_t)^{1-\theta}\right)^{1-\gamma}}{1-\gamma}\right] \tag{1}$$

subject to

$$P_tC_t + B_t \le W_tL_t + RB_{t-1} + P_tx_t - \tau\ . \tag{2}$$

Here B is borrowing, R and τ are the gross nominal interest rate and lump-sum taxes, both set to constants by government policy, C is consumption, and x is dividends from the firm, which the consumer is taken to own. Note that x is not a decision variable. The consumer treats it as an exogenous process. The consumer must also satisfy $B_t \ge -\bar{B}$, $C_t \ge 0$ and $L_t \in [0,1]$, though you can take all these constraints to be non-binding in equilibrium.

The government budget constraint requires that

$$B_t = RB_{t-1} - P_t\tau_t\ . \tag{3}$$

The representative firm solves

$$\max_{\{K_s,L_s,I_s,C_s,x_s\}} E\left[\sum_{t=0}^{\infty}\beta^t\Phi_tx_t\right] \tag{4}$$

subject to

$$P_tx_t \le P_tC_t - W_tL_t \tag{5}$$

$$C_t + I_t\cdot\left(1+\phi_t\frac{I_t}{K_{t-1}}\right) \le A_tK_{t-1}^{\alpha}L_t^{1-\alpha} \tag{6}$$

$$K_t \le I_t + (1-\delta)K_{t-1} \tag{7}$$

We have imposed market-clearing implicitly here in some cases by using the same symbols for firm and consumer problem variables.

We make the conventional fudge of assuming that, somehow, the firm's discount factor Φ_t, though taken as exogenous by the firm, exactly matches the stochastic discount factor that emerges from the consumer's problem (so that Φ_{t+1}/Φ_t is the ratio of marginal utilities of consumption in the two periods). This is what would emerge from complete asset markets, though we have of course not modeled these complete asset markets explicitly.[1]

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The exogenous processes A and ϕ are both i.i.d. and their logs have mean zero, variances σ_A^2 and σ_ϕ^2. We will assume $R = \beta^{-1}$, $\tau = 1$, $\beta = .95$, $\theta = .6$, $\gamma = 1.5,\ 2,$ or 5, $\alpha = .4$, and $\delta = .05$.

(a) Derive the first-order conditions for an optimum, including transversality, for the firm and for the consumer.

(b) Find the steady state of the model for each value of γ. [This can be done analytically. Start by finding K/L from the K and I FOC's. The steady state value of B is not uniquely determined, so you need to impose a value for it. Use $B = 19$.]

(c) For each value of γ, linearize the model about steady state and use `gensys.m` (or other software, if you prefer) to verify existence and uniqueness of a stable solution to the linearized system. [This system is big enough so that it is probably best to use numerical methods to find the linearization, though using analytic derivatives is certainly possible. The answer to the closely related problem on the last-year's-course web page shows (at the end) how, if you write a matlab function that evaluates the equations of the system, the numerical differentiation can be done in just a couple of lines of code.]

(d) For which values of γ, if any, is the simple stochastic permanent income theory conclusion that $E_t C_{t+1} = C_t$ approximately valid? The answer may depend on σ_A^2/σ_ϕ^2. (Note that looking at `gensys`'s `G1` alone may not give you the answer, since there are likely to be exact dependencies among variables. If all the impulse responses of C to ε and ν are nearly completely flat, then $E_t C_{t+1} = C_t$ is a good approximation. Why?)

(e) It is often argued that in order to guarantee a stable price level, monetary policy must increase the nominal interest rate in response to inflation, by enough to cause a real rate increase. This would imply in particular that a monetary policy that fixes the nominal rate produces a bad outcome. Is that true in this model? (We will be discussing this type of issue later, under the heading of the fiscal theory of the price level.)

[1] If you use the budget constraint (2) as given, this assumption means that $\Phi_t/\Phi_{t+1} = \lambda_t P_t/(\lambda_{t+1} P_{t+1})$, where λ_t is the Lagrange multiplier on (2) in the consumer's problem. If you divide the constraint through first by P_t, then the condition is instead that the ratio of the $\Phi's$ is the ratio of the λ's.