Capital Taxation Exercise Answers*

(a) Using transversality, show that equilibrium $\kappa = K/C$ is indeed constant in this model. Find κ as a function of the model parameters α, β, τ, and ψ.

FOC's for the household are

$$\partial C: \quad \frac{\theta}{C_t} = \lambda_t$$

$$\partial L: \quad \frac{1-\theta}{1-L_t} = (1-\psi)w_t$$

$$\partial K: \quad \lambda_t = \beta(1-\tau)R_{t+1}\lambda_{t+1}$$

$$\partial B: \quad \lambda_t = \beta\rho_t\lambda_{t+1}\,.$$

For the firm they are simply

$$\partial M: \quad \alpha M_t^{\alpha-1}L_t^{1-\alpha} = R_t$$

$$\partial L: \quad (1-\alpha)M_t^{\alpha}L_t^{-\alpha} = w_t\,.$$

The non-trivial transversality conditions are $\beta^t\lambda_t K_t \to 0$ and $\beta^t\lambda_t B_t \to 0$. Using the ∂C FOC and the ∂M FOC in the ∂K FOC gives us

$$\frac{1}{C_t} = \beta(1-\tau)\frac{\alpha K_t^{\alpha-1}L_{t+1}^{1-\alpha}}{C_{t+1}}\,.$$

Multiplying through by K_t and using the goods market clearing condition (or social resource constraint) $C_t + K_t = K_{t-1}^{\alpha}L_t^{1-\alpha}$ gives us the following difference equation in $\kappa_t = K_t/C_t$:

$$\kappa_t = \beta(1-\tau)\alpha(1+\kappa_{t+1})\,.$$

This is an unstable difference equation in κ, which makes κ explode toward $\pm\infty$ at the rate $(\alpha\beta(1-\tau))^{-t}$ unless κ remains constant at its steady-state value of

$$\bar{\kappa} = \frac{\beta(1-\tau)\alpha}{1-\beta(1-\tau)\alpha}\,.$$

We know that the explosive paths are impossible because negative K/C is technically infeasible and because, once we have used the ∂C FOC to substitute λ out of the K transversality condition, it says that $\kappa_t\beta^t \to 0$, which would be violated if κ exploded upward at the rate implied by our difference equation in κ above.

(b) Find steady state C and L as functions of the parameters, and from that find steady state period utility $\log\big(C_t^\theta(1-L_t)^{1-\theta}\big)$.

Using $\bar{k}$ to stand for steady-state K_t/L_t, we can see from the K FOC and M FOC (plus K-market clearing) that

$$\bar{k}^{1-\alpha} = \alpha\beta(1-\tau)\ .$$

It is a matter of straightforward algebra to show that

$$\frac{\bar{C}}{1-\bar{L}} = \frac{\theta(1-\psi)(1-\alpha)k^\alpha}{1-\theta}\,, \qquad (*)$$

where $k_t = K_{t-1}/L_t$. Using the SRC and our knowledge that κ is constant, we can write

$$K_t(\kappa^{-1}+1) = L_t k_t^\alpha\ .$$

Taking the ratio of these last two expressions, the k_t terms cancel, to leave us with

$$\frac{1}{(1+\kappa)(1-L_t)} = \frac{\theta(1-\alpha)(1-\phi)}{L_t(1-\theta)}\ .$$

This equation can be solved for a constant value of leisure,

$$1-L_t = \frac{1}{1+\dfrac{(1+\kappa)\theta(1-\psi)(1-\alpha)}{1-\theta}}\ . \qquad (**)$$

Since steady state period utility is the log of $(1-\bar{L})(\bar{C}/(1-\bar{L})^\theta$, the desired result follows easily from $(*)$ and $(**)$.

(c) Find the derivatives of steady-state period utility with respect to τ and ψ at the point $\tau=\psi=0$.

Note that $\bar{k}$ and $\bar{\kappa}$ depend on τ, but not on ψ. Steady state period utility is

$$\theta\log(1-\psi) + \frac{\theta}{1-\alpha}\log(1-\tau) + \log(1-\alpha\beta(1-\tau)) - \log\big((1-\theta)(1-\alpha\beta(1-\tau)+\theta(1-\psi)(1-\alpha)\big)\,,$$

plus terms that don't depend on ψ or τ. Differentiating this and using the fact that we are evaluating at $\tau=\psi=0$, we conclude that the derivative of utility with respect to ψ at the point that interests us is

$$-\theta\left(1-\frac{1-\alpha}{(1-\theta)(1-\alpha\beta)+\theta(1-\alpha)}\right)\ .$$

The denominator of the fraction in this expression is a weighted average, with positive weights summing to one, of $1-\alpha$ and $1-\alpha\beta$. Since $\beta<1$ and $0<\alpha<1$, this weighted average is positive and larger than $1-\alpha$, from which it follows that the whole expression is negative.

Now we can go through a similar calculation differentiating steady state period utility with respect to τ. The result is

$$\frac{-\alpha\theta}{1-\alpha}\left(1-\frac{(1-\alpha)^2\beta}{(1-\alpha\beta)\big((1-\alpha\beta)(1-\theta)+\theta(1-\alpha)\big)}\right) . \tag{1}$$

The same kind of argument about the weighted average in this expression leads to the same conclusion we obtained for ψ, that increases in τ from $\tau = 0$ produce negative effects on steady state utility.

(d) Find time-0 utility (i.e. the value of the consumer objective function) as a function of model parameters and initial capital stock K_{-1}.

The fixed values of κ and L along any given solution path let us write

$$K_t(1+\bar{\kappa}^{-1}) = K_{t-1}^{\alpha}\bar{L}^{\alpha} ,$$

so that $\log K_t$ follows a stable first-order difference equation, converging at the rate α^t to its steady-state value $\bar{K}$. Since $\kappa_t = K_t/C_t$ is constant, C also follows a path of exponential convergence to its steady state value. Thus it is not too hard to evaluate the objective function analytically. It consists of a discounted sum of $(1-\theta)\log(1-L_t)$, which is constant, plus a discounted sum of $\theta \log C_t$, which is a constant (steady-state C) plus a term that converges exponentially to zero. Introducing a tax affects the objective function by changing $\bar{L}$, $\bar{K}$, and $\bar{\kappa}$, and thereby $\bar{C}$ and C_0. Using these ideas, we get that discounted utility is

$$\frac{1-\theta}{1-\beta}\log(1-\bar{L}) + \frac{\theta}{1-\alpha\beta}\log(C_0/\bar{C}) + \frac{\theta}{1-\beta}\log \bar{C} .$$

Since we know the steady state value of $\bar{L}$, $\bar{\kappa}$, and $\bar{k}$ as functions of parameters from calculations above, we can substitute into this expression to get discounted utility as a function of parameters and K_{-1}.

(e) Find the derivatives of time-0 utility with respect to τ and ψ at the point $\tau = \psi = 0$. If it makes it easier, you can also assume and $K_{-1} = \bar{K}$, where $\bar{K}$ is the steady-state value of the capital stock when $\tau = \psi = 0$.

With several clean sheets of paper and the expression for utility in terms of parameters in front of you, it is quite possible to find these derivatives, and they are in fact zero, for both ψ and τ. It took me longer than I expected to get the algebra worked out, though, and I don't see a way to transcribe it all here in a way that will be useful to anyone. There may be a clever way to show that the derivatives without the brute force algebra I used, and if anyone has come up with one, I'll post it.

(f) Explain whatever qualitative differences you find between the response of period utility and the response of time-0 utility to tax changes.

Imposing either kind of tax result in an increase in current utility and a reduced future utility. Since we begin in an efficient equilibrium, where consumption and leisure trade off technologically at the same rate that they do in utility, and consumption and leisure at different dates trade off technologically at the same rate that they do in utility, the small movements along the intertemporal production possibility frontier induced by small taxes have no first-order effects, despite their possible large effects on steady-state utility.