Current Account Exercise Answers*

1. Linearize the bonds-only model from the lecture about the $B = 0$, $C_1 \equiv C_2 \equiv \bar{Y}$ steady state, assuming $U(C_i(t))$ is of the CRR form $C^{1-\gamma}/(1-\gamma)$. Solve the resulting linear model, verifying existence and uniqueness, either by hand or using the computer with $\beta = .95$, $\bar{Y} = 1$, $\gamma = 2$, and $Y_i(t)$ i.i.d. across both i and t. Solve also for the case where instead $\Delta Y_i(t)$ is i.i.d. across i and t. Discuss the difference.

Though this problem can of course be done numerically, it is also fairly easy to solve analytically. The deterministic steady state will, from the agent FOC's, have $\rho = \beta^{-1}$. We start with the set of equations

$$
\begin{aligned}
C_1(t)^{-\gamma} &= \beta\rho_t E_t\left[C_1(t+1)^{-\gamma}\right] \\
C_2(t)^{-\gamma} &= \beta\rho_t E_t\left[C_2(t+1)^{-\gamma}\right] \\
B_t + C_1(t) &= \rho_{t-1}B_{t-1} + Y_1(t) \\
-B_t + C_2(t) &= -\rho_{t-1}B_{t-1} + Y_2(t)\ .
\end{aligned}
$$

(Note that we have used the market clearing condition $B_1 + B_2 = 0$ to replace B_1 and B_2 by B and $-B$.) We can simplify a bit by replacing the last equation with the sum of the last two, the SRC $C_1(t)+C_2(t) = Y_1(t) + Y_2(t)$. (Note that we should not fall into the trap of thinking we can use the SRC plus the two Euler equatons as a self-sufficient equation system in 3 unknowns. The only stability system we have applies to B. If we omit B and the stability condition on it we will obtain a spurious indeterminacy result.) If we do that, then linearize about steady state with variables ordered as $y_t = [C_1, C_2, \rho, B]$, we get for the matrices in the canonical form $\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi\eta_t$

$$
\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 & \beta/\gamma & 0 \\ 0 & 1 & \beta/\gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{bmatrix}
$$

$$
\Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
$$

At this point one option is to proceed straight to `gensys`. Doing so produces

$$G1 = \begin{bmatrix} 0.0000 & -0.0000 & 0.0000 & 0.0526 \\ 0.0000 & -0.0000 & 0.0000 & -0.0526 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

$$\texttt{impact} = \begin{bmatrix} 0.5250 & 0.4750 \\ 0.4750 & 0.5250 \\ -1.0526 & -1.0526 \\ 0.4750 & -0.4750 \end{bmatrix}$$

$$\texttt{ywt} * \texttt{fwt} = \begin{bmatrix} 0.0238 & -0.0238 \\ -0.0238 & 0.0238 \\ 1.0526 & 1.0526 \\ -0.0238 & 0.0238 \end{bmatrix}$$

$$\texttt{ywt} * \texttt{fmat} * \texttt{fwt} = \begin{bmatrix} 0.0226 & -0.0226 \\ -0.0226 & 0.0226 \\ -0.0000 & -0.0000 \\ -0.0226 & 0.0226 \end{bmatrix}$$

$$fmat = \begin{bmatrix} -0.0000 & 0.0000 \\ -0.0000 & 0.9500 \end{bmatrix}$$

From this we see that, in the case of i.i.d. Y's, the solved system (with deviations from steady state indicated by a "d" prefix) is

$$\begin{aligned} dC_1(t) &= .0526dB(t-1) + .525dY_1(t) + .475dY_2(t) \\ dC_2(t) &= -.0526dB(t-1) + .475dY_1(t) + .525dY_2(t) \\ d\rho(t) &= -1.0526\big(dY_1(t) + dY_2(t)\big) \\ dB(t) &= dB(t-1) + .475\big(dY_1(t) - dY_2(t)\big) \end{aligned}$$

Endowment shocks from the two sources impact the two agents almost, but not quite, equally. Debt follows a random walk driven by the difference in the two endowment processes. It eventually will get far from its "steady state" of zero, and will then start to have a large impact on the relative consumptions of the two agents. Of course by this time the linear approximation, which is valid only in the neighborhood of $B = 0$, will have broken down in accuracy. (One can relinearize around a new B value, to get some idea of how the system behaves away from $B = 0$. The unit root in B does not go away. The mean consumption of the agent that is in debt goes down and that agent's consumption variance goes *up*. The variance of the changes in debt also goes up.

The linearized system thus has no tendency to stay away from the no-Ponzi bounds on debt.)

Using the forward part of the solution from `gensys`, one can also get the answer to the second part of the question. The `ywt*fwt` term is the coefficient matrix on the first expected future value of the Y vector, the `ywt*fmat*fwt` the second term, etc. The `fmat` matrix determines the rate of decay of the future coefficients, which is clearly at the rate $.95^t$. For the case you are asked to consider $E_t[dY_i(t+s) = dY_i(t)$, all s, t, i. The absolute value of the sum of coefficients on future values of Y in each equation except the third (the ρ equation) is $.0238/.05 = .475$. (This is not the answer you get by doing the division, but that is because of rounding error in the 4-decimal-place figures.) It is not hard to see therefore that the solution in this case reduces to $dC_i(t) \equiv dY_i(t)$, all t, i, and therefore $dB(t) \equiv 0$. In other words, because all shocks to endowments are permanent, the opportunity to borrow and lend is not used at all and the competitive equilibrium is the same as with autarchy.

Thus we see that with i.i.d. Y shocks borrowing and lending substantially alters the autarchy equilibrium and brings us close to the planner's optimum (at least for a while, until B drifts away from 0). But this depends on shocks being non-persistent. With random walk Y's, every shock is permanent and borrowing and lending can therefore do nothing to improve on autarchy.

It may be worthwhile to go through how an analytic solution would work here, as on an exam one needs an analytic solution. Also, what follows suggests some generally applicable tricks that were not discussed in lecture.

The Γ_0 for this problem is singular, as can be seen directly from the fact that its 3rd column is zero. Thus one cannot simply multiply the system through by Γ_0^{-1} to achieve the form that was discussed in class. It is possible to find a linear combination of the rows of Γ_0 that is zero, then use that linear combination as one of the equations of the system, but with its time subscripts shifted. (There are some complications in handling the error term.) There is an easier way, though.

If Γ_0 were invertible, we could find the eigenvalues of $\Gamma_0^{-1}\Gamma_1$ from the equation $\left|\Gamma_0^{-1}\Gamma_1 - \lambda I\right| = 0$. Of course we could find the same λ's as the roots of the equation $|\Gamma_1 - \lambda\Gamma_0| = 0$. It turns out that we can use this equation to find unstable roots of systems even when $|\Gamma_0| = 0$.

So in this case we find the roots of

$$\left|\begin{bmatrix} 1-\lambda & 0 & \beta/\gamma & 0 \\ 0 & 1-\lambda & \beta/\gamma & 0 \\ -\lambda & -\lambda & 0 & 0 \\ -\lambda & 0 & 0 & \beta^{-1}-\lambda \end{bmatrix}\right| = 0\,.$$

These turn out to be β^{-1}, 1, and 0. The unstable one is β^{-1}. We can now look for a vector x such that

$$\beta^{-1} x\Gamma_0 = x\Gamma_1\,. \tag{$*$}$$

Such an x is what is called a **left generalized eigenvector** of the pair of matrices Γ_0, Γ_1. If we normalize one element of x to 1, say its first element, and set $x = \begin{bmatrix} 1 & a & b & c \end{bmatrix}$, we get from $(*)$ a set of equations to solve for a, b and c. These result here in $a = -1, b = 1-\beta, c = 2(\beta - 1)$. The unstable component of the system is then the single equation

$$\begin{aligned} w_t = x\Gamma_0 y_t &= x\Gamma_1 y_{t-1} + x\Psi dY_t + x\Pi\eta_t \\ &= \beta^{-1} x\Gamma_0 y_{t-1} + x\Psi dY_t + x\Pi\eta_t = \beta^{-1} w_{t-1} + x\Psi dY_t + x\Pi\eta_t\,. \end{aligned} \tag{§}$$

When $Y_i(t)$ is i.i.d., the unique stable solution of this equation implies $w_t \equiv 0$, $x\Psi dY \equiv -x\Pi\eta$. The first of these reduces to

$$dC_1(t) - dC_2(t) = 2(\beta^{-1} - 1)dB(t)\,, \tag{†}$$

and the second to

$$\eta_1(t) - \eta_2(t) = (1-\beta)(dY_1(t) - dY_2(t))\,.$$

By substituting the two budget constraints into (†) to eliminate the C's, we obtain the random-walk equation for B as a function of Y's, and from the B solution thus generated we can solve for the time paths of the two C_i's as functions of B and Y_i's. To solve for ρ we add the first two equations and subtract the third, leaving us with an equation in lagged $d\rho$, lagged dB, $dY_1 + dY_2$, and $\eta_1 + \eta_2$. Because the lagged terms are known at t, the two date-t stochastic terms must cancel out, so the equation becomes one for $d\rho$. To find the solution for the random-walk Y case, one would use the forward solution to (§), but I omit the details here.

2. Show that the planner's allocation $C_1(t) \equiv C_2(t) \equiv (Y_1(t) + Y_2(t))/2$ is not a competitive equilibrium in the bonds-only economy with randomness in Y, even if stochastic shocks are small and become zero after a finite number of periods.

If we subtract the two budget constraints we arrive at

$$B_t = \frac{C_2(t) - C_1(t)}{2} + \rho_{t-1}B_{t-1} + \frac{Y_1(t) - Y_2(t)}{2}\,.$$

In the planner's allocation, $C2 \equiv C_1$, so the first term on the right of this expression is zero, and we conclude that B satisfies an unstable difference equation driven by the exogenous disturbance $Y_1 - Y_2$. This implies that B explodes up or down at the rate β^{-t} unless $B \equiv 0$ and $Y_1 - Y_2 \equiv 0$. Thus the only way for the planner's allocation to be an equilibrium is for initial B to be zero and for the $Y(t)$'s at *all* dates to be distributed in such a way that $Y_1(t) = Y_2(t)$ with probability one, which is certainly not true if they are independent.

3. Determine what emerges as a competitive equilibrium in the stock-trading economy when it starts at $t = 0$ with $S_i(-1) = 0$ for $i = 1, 2$. Assume that U is CRR. (without that, the Y_i-stocks don't complete the market. [Hint: It won't be $C_1 \equiv C_2 \equiv (Y_1 + Y_2)/2$, but it will be efficient — though perhaps not just.]

Assume, as the problem statement certainly suggests, that the stock market equilibrium is efficient. That means that at every date $U'(C_1(t)) = \theta U'(C_2(t))$ for some fixed θ. Otherwise it would be possible to increase total utility by transferring consumption from 1 to 2 at some date and from 2 to 1 at another. With CRRA utility, this means $C_1(t)^{-\gamma} = \theta C_2(t)^{-\gamma}$ at all t, and thus that $C_1(t) = \kappa C_2(t)$ at all t, where $\kappa = \theta^{-1/\gamma}$. Suppose we can support this equilibrium with an asset portfolio as in the equal-C allocation, such that each agent is just consuming her income. This would require that

$$C_1(t) = \frac{\kappa}{1+\kappa}(Y_1(t) + Y_2(t)) = (1 - S_1)Y_1(t) + S_2(t)Y_2(t)\,.$$

Equating coefficients on the two random variables $Y_1(t)$ and $Y_2(t)$, we conclude that $S_2 = \kappa/(1+\kappa)$ and $S_1 = 1/(1+\kappa)$. But if this is to be the ratio of C's in all periods, it must also be the ratio in the first period, which the agents enter with no holdings of stock. In that first period, they will have to trade assets if their endowments fail to arrive in the ratio κ. That is, we will have to have in the first period

$$C_1(1) = \frac{\kappa}{1+\kappa}(Y_1(1) + Y_2(1)) = Y_1(1) + Q(1)(S_1 - S_2) = Y_1(1) + Q(1)\frac{1-\kappa}{1+\kappa}\,. \qquad (\ddagger)$$

With i.i.d. Y's, as we discussed in class, the asset price will be the same for both assets and will depend only on $Y_1(t) + Y_2(t)$. Q is found by solving the Euler equations forward, and therefore does not depend on any way on the S_i's. Thus $Q_i(t) = \bar{Q}(Y_1(t) + Y_2(t))^{\gamma}$ and $(\ddagger)$ can be solved at time 1 for κ as a function of the initial Y_1 and Y_2. In fact, we have

$$\kappa = \frac{Q + Y_1}{Q + Y_2}\,.$$

Thus as we would expect, the agent with the larger initial endowment gets the larger permanent share of the endowment stream, but since Q

will be on the order of $Y/(1-\beta)$, i.e. much bigger than the Y's, the inequality will be modest, with the permanent ratio of consumptions much closer to 1 than the initial ratio of endowments. It is also worth noting that for any positive initial values of the Y's, there is a positive κ that satisfies the model's equilibrium conditions. It is easily checked that with κ determined this way by agent 1's budget constraint, agent 2's budget constraint is also satisfied automatically. Of course the FOC's are satisfied because we have chosen $\bar{Q}$ to guarantee that.

For the constraints and FOC's of the bonds-only and stock-trading models, see the notes to Lecture 7, which are on the course web site.