Econ. 511b Spring 1996 C. Sims

Answers to 511b Final

Note: These answers are considerably more complete than what was expected on the exam. In some cases they explore more than one way to answer the same question or, as in 2.v, give long precise answers where a shorter qualitative one would have sufficed.

1.

i) The non-stochastic steady state has $r_t = \bar{r}$, $p_t = \bar{p}$, and then, from (1) with time subscripts suppressed, $\bar{Q} = \dfrac{\bar{p}}{\bar{r}-1}$. If we log-linearize, equations (2) and (3) are already in the appropriate form, though we might want to drop the constants and write them in terms of deviations from steady state. Equation (1), linearized is

$$dq_t + (\bar{r}-1)dp_t = \bar{r}dq_{t-1} + \bar{r}dr_{t-1} + h_t \ , \tag{A.1}$$

where h_t is an endogenous expectational error, $q = \log Q$, $p = \log p$, and d indicates deviation from steady state. In matrix notation, the full system is then

$$\begin{bmatrix} 1 & 0 & \bar{r}-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dq_t \\ dr_t \\ dp_t \end{bmatrix} = \begin{bmatrix} \bar{r} & \bar{r} & 0 \\ 0 & g & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} dq_{t-1} \\ dr_{t-1} \\ dp_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_t \\ x_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} h_t \ . \tag{A.2}$$

ii) There are several possible approaches here. The low-tech approach notes that (A.1) is a one-variable system with two exogenous and one endogenous "shocks". This holds because equations (2) and (3) determine the stochastic processes for r and p by themselves. The one-equation system does not quite fit in our standard framework from the notes, because the dr_{t-1} term has a lag. It could be put in to our framework precisely by introducing $z_t = (1-\bar{r})dp_t + \bar{r}r_{t-1}$, but this is not necessary here. (A.1) has just one root, $\bar{r}$, which is presumably greater than one and is in any case ruled out by assumption as a potential rate of growth for Q. Thus if it has a solution, it is the forward solution given by

$$q_t = E_t\left[\sum_{s=1}^{\infty} \bar{r}^{-s}\left((\bar{r}-1)dp_{t+s} - \bar{r}dr_{t+s-1}\right)\right] \tag{A.3}$$

There is no problem with existence or uniqueness, because the equation has one endogenous shock ready to absorb and be defined by the exogenous shock at each date. Using (2) and (3) we see that $E_t dr_{t+s} = g^s dr_t$ and $E_t dp_{t+s} = f^s dp_t$. Substituting into (A.3) we get

$$dq_t = \frac{\bar{r}-1}{\bar{r}-f} f \cdot dp_t - \frac{\bar{r}}{\bar{r}-g} dr_t \ . \tag{A.4}$$

The high-tech approach uses the matrix form of the system and starts extracting eigenvectors and eigenvalues. It is not necessarily harder, but if one followed that approach in a mechanical way (e.g., extracting all *right* eigenvectors of $\Gamma_0^{-1}\Gamma_1$, then inverting that to find left eigenvectors) it was easy to get bogged down. Here is a working version of a high-tech approach. The equation defining the generalized eigenvalues of Γ_0, Γ_1 is

$$\begin{vmatrix} \bar{r}-l & \bar{r} & \bar{r}-1 \\ 0 & g-l & 0 \\ 0 & 0 & f-l \end{vmatrix} = 0 \qquad \text{(A.5)}$$

It is easy to see that this has roots of $\bar{r}$, g and f. The left eigenvector c associated with the first, unstable, root must satisfy

$$\bar{r}c'\Gamma_0 = c'\Gamma_1 \ . \qquad \text{(A.6)}$$

Letting the elements of c be x, y, z, this produces the equations

$$\begin{aligned} \bar{r}x &= \bar{r}x + fz \\ \bar{r}y &= \bar{r}x + gy \\ \bar{r}\cdot(\bar{r}-1)x + \bar{r}z &= fz \end{aligned} \qquad \text{(A.7)}$$

Normalizing x at 1, this gives us

$$c' = \left[1,\ \bar{r}/(\bar{r}-g),\ -\bar{r}\cdot(\bar{r}-1)/(\bar{r}-f)\right]. \qquad \text{(A.8)}$$

Then applying the requirement (for non-explosiveness of the solution) that

$$c'\Gamma_0 dy_t = 0 \qquad \text{(A.9)}$$

it is easy to check that we arrive exactly at (A.4).

iii) One approach to this section is to consider values of f and g outside the (0,1) range. Of course values much above 1 would imply explosiveness violating our maintained assumption. But negative values less than one in absolute value are quite possible. One can see directly from (A.4) that the effect of dividends on stock price reverses sign when f changes sign, whereas there is no corresponding way to change the negative effect of an increase in the interest rate. This is because high current dividends have no direct positive effect on the stock price, which (as we see from (A.3)) depends only on *future* dividends. When serial correlation is negative, the high current dividends imply a lower dividend next period, which accounts for the sign reversal. There is a direct current negative effect of high interest rates on stock price, however, and this is only reduced, not eliminated, by negative serial correlation. From (A.3) we can see that it would be possible to have sign reversal, but it would require complicated high order serial correlation, in which increased r now implies a long period of low r later.

2.

i) The social resource constraint is found by using (5) and (6) to eliminate t and B, yielding

$$C_t + K_t = A(1-g\,)K_{t-1} \tag{A.10}$$

Several people essentially repeated the mistake on the mid-term that many people had made, and that we discussed in class – they left the 1-*g* factor out of the equation, ignoring the fact that *G* requires resources and is said to be determined exogenously. Another fairly common mistake was to make no use of the middle section of (6), arriving at a constraint that still contains *G*. This is not wrong, but it seems to give the government an extra decision variable *G* which is not really free. If the $gK_{t-1} = G_t$ were imposed as a second social resource constraint, the resulting problem would give the right answer. The FOC's, which most people found correctly, reduce to (after Lagrange multipliers are eliminated)

$$\frac{1}{C_t} = bE_t\left[\frac{A\cdot(1-g\,)}{C_{t+1}}\right]. \tag{A.11}$$

Quite a few people stopped after deriving (A.11), but this is not a solution. It does not tell us how to choose *C* at a given date. Since the problem states that the only source of uncertainty is future government actions, for the government's own optimization here we can drop the *E* operator in (A.11) and convert it to linear form. Then (A.11) and (A.10) form a linear system with one unstable root that could be analyzed by our standard methods. In fact, no one did this. Most guessed correctly that the solution would have C_t proportional to K_t or K_{t-1} and solved for the constant of proportionality. If $C_t = yK_t$, then from (A.10) we have

$$(1+y^{-1})C_t = A\cdot(1-g\,)y^{-1}C_{t-1}\;. \tag{A.12}$$

For this to be compatible with (A.11), we need to have

$$bA\cdot(1-g\,) = \frac{A\cdot(1-g\,)}{1+y},\;\therefore\; y = b^{-1}-1\;. \tag{A.13}$$

ii) The private agent's FOC's reduce to

$$\frac{1}{C_t} = bE_t\left[\frac{A(1-t_{t+1})}{C_{t+1}}\right], \tag{A.14}$$

$$\frac{1}{C_t} = br_tE_t\frac{1}{C_{t+1}}\;. \tag{A.15}$$

If the tax rate is fixed, then (A.14) matches (A.11) when $\bar{t} = g$. But we still need to verify that the private budget constraint matches the social resource constraint and that the government budget constraint, including the bound on the rate of growth of debt, is satisfied. When $\bar{t} = g$, the latter equality in (6) implies that $B_t = r_{t-1}B_{t-1}$, all *t*. Since we start with zero debt, this means we never have any debt, so certainly *B* is not exploding. Also we see by looking at (5) that with *B*=0, the private resource constraint looks like the social constraint. This does not mean it *is* the social constraint (a subtlety no one observed under exam time pressure), because at each date private agents see themselves as free to choose non-zero debt. Because in equilibrium $r \equiv \bar{t}$, they are indifferent between

B and K, and the government happens not to issue any B. Nonetheless, since the FOC (A.14) matches the social FOC, and since the condition for preventing K from exploding (based on the social constraint or the private constraint with the equilibrium condition B=0 imposed) matches, the two solutions are in fact the same.

This result emerges here because there is an externality. Increasing K forces the government to increase G, imposing a real resource cost that private decision makers do not take account of. Setting $\bar{t} = g$ makes them "see" the external social cost of their investments.

iii) If t is constant, we know from (A.14) that the only way to make the solution match the social optimum is to have $t = g$. But then from the private FOC's $r = A \cdot (1-g)$ and the government budget constraint is

$$B_t = A \cdot (1-g) B_{t-1} \ . \tag{A.16}$$

Obviously with $B_{-1} > 0$, B does explode at a rate equal to the interest rate, and this is ruled out by transversality. (The problem said growth at a rate "faster" than the interest rate is ruled out, but this is not correct – even growth at a rate equal to the interest rate is ruled out.) Some people got tangled up over the possibility that $r = A \cdot (1-g) < 1$. This is indeed possible in principle, and transversality still does rule out growth – in this case shrinkage – at that rate. In this case consumption shrinks at the rate br, so that even though B is shrinking exponentially, its ratio to C is growing exponentially, and this is suboptimal.

iv) If the government can announce tax rates in advance and be believed, it should announce at time zero that $t_0 = r_{-1} B_{-1} / AK_{-1}$, $t_t = g$ for all $t > 0$. This will eliminate all debt in the first period and set the economy on the socially optimal track. Only t_t for $t > 0$ enters the private FOC's, so they will all be as in the social planner's problem. We have already analyzed the situation for time t=1 onwards, where t has been announced to be, and is, constant, with $B = 0$. The only question is whether the high taxes at time 0 might make individuals see themselves as having a shrunken budget set and hence behave suboptimally. But here we are in a "Ricardian" situation. Individuals can see that the high taxes at time zero correspond to lower taxes at all future dates, so their overall budget set is left unchanged by the high initial taxes. Because t_0 is a tax on K_{-1}, which is inelastically supplied, it has no distorting effects.

v) Here, with agents having "static expectations," t_0 does have effects on C_0. We found in the social planning problem that $C_t / K_t = (1-b)/b$. This static expectations private problem has the same structure, because expected rates of return on the two assets K and B are identical, so we arrive at

$$C_t / (K_t + B_t) = (1-b)/b \ . \tag{A.17}$$

Also (A.14) and (A.15) imply $r_t = AE_t(1-t_{t+1})$, but because agents believe next period's tax rates will match this period's, we get $r_t = A \cdot (1-t_t)$. Then from the private budget constraint we obtain

$$\frac{C_t}{1-b} = A\cdot(1-t_t)K_{t-1} + A\cdot(1-t_{t-1})B_{t-1}\,. \qquad \text{(A.18)}$$

From (A.18) it is clear that the current tax rate affects *C*. You got substantial credit for getting to this point, then observing that generally the government is going to be able to manipulate *C* here with the tax rate and therefore should not be expected to stay at the suboptimal *C* generated by a constant tax rate. The messy part, which few attempted, is checking whether actually obtaining the social optimum is consistent with the government budget constraint.

From (A.17) it is clear that inducing consumers to consume the socially optimal amount is equivalent to inducing them to hold no debt. This is because the ratio of *C* to total wealth in (A.17) is what is socially optimal for a ratio of *C* to *K*, according to (A.13). Thus if initial debt is not so large that this is impossible, optimal taxation consists of setting the first period tax rate so that consumers cash in all their debt to pay it, then thereafter setting $t = g$. This looks exactly like the optimal policy with perfect foresight, and even the initial tax rate is exactly the same. This equivalence is an artifact of the logarithmic utility function, which makes the proportion of current wealth saved invariant to the rate of return. If initial debt is larger than can be absorbed by a finite initial tax rate on *K* less than 1, it will still generally improve utility to have taxes high enough to reduce the ratio of debt to *K*. Proving this in detail was not expected, but you might have noted that it is a matter of verifying that the tradeoff of current consumption sacrifice for future gains is favorable.

3. This question was harder than I meant it to be. I thought the first case was a straightforward repeat of what was done in class and in the notes, though I knew the latter two were harder. It turns out that even the first case was not simple, though, with no hint in the problem statement to look for it, no one recognized what made it hard: that the usual transversality argument that rules out unbounded real money balances does not apply directly to interest-bearing debt.

$r_t = \bar{r}$, $B_t = \bar{B}$: Most people got most of the way through an analysis of this problem that would have been correct if transversality worked the way people (incorrectly) assumed. You got nearly full credit for this, and extra points if you indicated any uncertainty about whether transversality rules out unbounded B/P. The FOC's for this problem reduce to

$$\frac{1-gv_t^2}{B_t v_t\cdot(1+2gv_t)} = br_t E_t\left[\frac{1}{B_{t+1}v_{t+1}\cdot(1+2gv_{t+1})}\right]. \qquad \text{(A.19)}$$

Under the policy of this section, this becomes

$$\frac{1-gv_t^2}{v_t\cdot(1+2gv_t)} = (1-gv_t^2)Z_t = b\bar{r}E_t Z_{t+1}\ . \qquad \text{(A.20)}$$

Just as in the notes, *Z* here is a monotone decreasing function of *v*. When $\bar{r}$ is not chosen too large (i.e., $\bar{r} < b^{-1}$), there will be a value $\bar{v}$ of *v* that satisfies $1-g\bar{v}^2 = b\bar{r}$, and (A.20) is solved by a *v* sequence that remains fixed at this $\bar{v}$. Since (A.20) is almost the same difference equation discussed in the notes, the same arguments used there tell us that it is unstable, so that if v starts out above $\bar{v}$ it grows arbitrarily large with non-zero probability. This is impossible because it would eventually make the left-hand side of (A.19) negative, so that the equation could not hold. If v starts out below $\bar{v}$, it gets arbitrarily close to zero with non-zero probability. Though a few hedged on this, and got credit for doing so, most people assumed that v arbitrarily close to zero, because it implies B/P arbitrarily large, is inconsistent with individual optimization, just as is arbitrarily large M/P in the model discussed in class. But for M/P it is feasible for an individual to "eat" some of current real balances now and never to restore the reduced real balances. This is not true for B/P. Since large B/P implies small P with this policy, the one-period GBC

$$\bar{B} + t_t P_t = \bar{r}\bar{B}$$

tells us that real taxes t must grow in direct proportion to P's decline. Since endowment income Y is i.i.d., the only source of income that grows fast enough to keep up with taxes, which the individual perceives as exogenously fixed, is interest on the debt. The individual therefore can never afford to eat without replacement any of current government debt holdings B/P, no matter how big the debt holdings may grow. Doing so would eventually leave the consumer unable to finance non-zero current consumption.

Thus the correct answer, which nobody got, is that this policy leaves the price level indeterminate. Though an equilibrium with stable velocity exists, so do "virtuous circle" deflationary equilibria, in which the price level steadily drops and the economy converges toward satiation in real balances. Because these equilibria have higher real balances than does the constant-velocity equilibrium, they deliver higher welfare as well. (The social cost of real balances is zero in this model.)

$r_t = \bar{r}$, $t_t = \bar{t}$: With this policy, the B_t terms in (A.19) don't drop out, so it is not usable as a single difference equation in a single unknown. Some people guessed correctly that there might be a constant-velocity equilibrium for this economy, though as I remember no one fully and correctly characterized it. With v constant, (A.19) becomes

$$(1 - g\bar{v}^2) = b\bar{r}E_t\left[\frac{B_t}{B_{t+1}}\right]. \tag{A.21}$$

Dividing the GBC through by B_t gives us

$$1 + \frac{\bar{t}\bar{v}}{C_t} = \frac{\bar{r}B_{t-1}}{B_t}, \tag{A.22}$$

which, together with (A.21), implies

$$1 + E_{t-1}\left[\frac{\bar{t}\bar{v}\cdot(1+g\bar{v})^2}{Y_t}\right] = \frac{1-g\bar{v}^2}{b}. \tag{A.23}$$

Since Y is i.i.d., this can be solved for $\bar{v}$ in terms of $\bar{t}$, $\bar{r}$ and $E[1/Y_t]$. With $0 < b < 1$, it is guaranteed to have a solution. Then (A.22) will uniquely determine B_t at each t from initial conditions. Then we can derive P_t from

$$\bar{v} = \frac{P_t Y_t}{(1+g\bar{v})B_t} \quad . \tag{A.24}$$

Thus there is indeed a constant-velocity solution that satisfies the social resource constraint, the GBC, and the private FOC. Note that with this policy there is no limit on *r*. If the government sets a high $\bar{r}$, the equilibrium involves steady inflation.

Proving that this equilibrium is unique, which I think is possible, is much harder. I didn't see a way to reduce the analysis to one dimension, and a multidimensional argument is beyond what you could have been expected to do on an exam.

$B_t = \bar{B}$, $t_t = a + bB_t/P_t$: Some postulated that this also would lead to a constant-velocity equilibrium, but it does not. Once again our standard difference equation (A.19) fails to reduce to one dimension, this time because it contains r_t even after the *B*'s have canceled out. But if *v* is constant, the equation becomes

$$1 - g\bar{v}^2 = br_t \quad , \tag{A.25}$$

which fixes *r* at a constant value also (though we still don't know which constant value for either *v* or *r*). The GBC becomes

$$aP_t = (\bar{r} - 1 - b)\bar{B} \quad . \tag{A.26}$$

Thus *P* is a constant. But with *v*, *P*, and *B* constant, *C* is implied also to be constant, which is impossible according to the social resource constraint $C_t \cdot (1+g\bar{v}) = Y_t$ so long as the exogenous endowment varies at all. So if there is an equilibrium with this policy, it involves a time varying velocity.

We can be sure that equilibrium with this policy, if it exists, delivers a unique price level at every date. This is because the GBC, even without the fixed *r* implied by fixed *v*, is

$$aP_t = (r_{t-1} - 1 - b)\bar{B} \quad . \tag{A.27}$$

At each date *t*, including in particular $t = 0$, (A.27) determines a unique P_t that is in fact known one period in advance. I don't know whether this unique initial *P* corresponds to a full equilibrium or not, however. Again it appears that there is no simple way to reduce the model to one dimension, and a multivariate analysis was too much to carry out on an exam.