Preliminaries

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Definitions

σ-field A collection $\mathcal{F}$ of subsets of S satisfying

- $S \in \mathcal{F}$;
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
- $A_i \in \mathcal{F}$, $i = 1 \ldots, \infty \ \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

measure A function $\mu : \mathcal{F} \mapsto \mathbb{R}^+$, where sF is a σ-field, satisfying

- $\mu(\emptyset) = 0$;
- $(\forall A \in \mathcal{F}) : \mu(A) \geq 0$;
- If $A_i \in \mathcal{F}$, $i = 1, \ldots, \infty$ and $A_i \cap A_j = \emptyset$, all $i \neq j$, then $\mu(\bigcup A_i) = \sum \mu(A_i)$.

probability measure A measure for which $\mu(S) = 1$.

Generating σ-fields

Theorem 1. *If $\mathcal{E}$ is any collection of subsets of S, there is a uniquely defined σ-field $\mathcal{F}$ that is the smallest σ-field containing $\mathcal{E}$.*

This gives us a way to generate σ-fields. For example, we can take E to be all the open (or all the closed) subsets of S. The minimal σ-field containing all the open sets is called the **Borel field**. The probabilities that you have seen before are probably all, or mostly, defined on the Borel σ-field of Euclidean space $\mathbb{R}^k$.

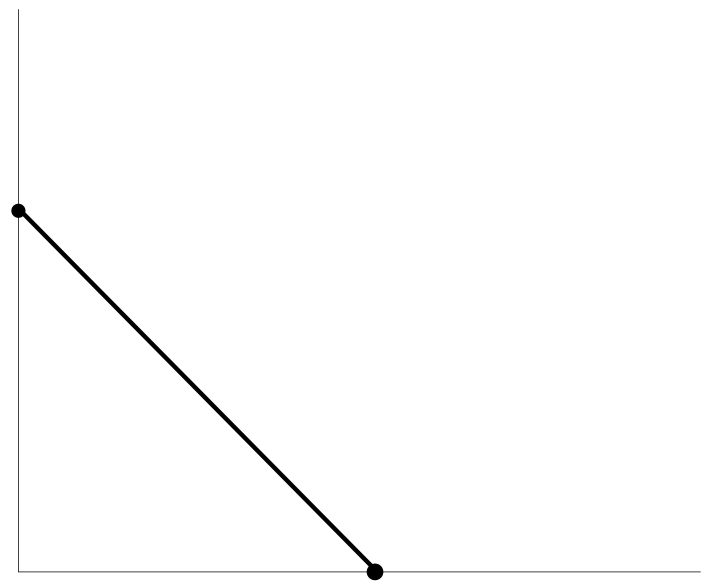
Generating measures

- Not as easy, generally. In particular, having a $\mu(A)$ defined for every $A \in \mathcal{E}$ is not always enough to determine a unique measure, on the σ-field generated by $\mathcal{E}$.

- **Lebesgue measure**. Defined on the Borel field of $\mathbb{R}^k$ (plus, as a technical addendum, all subsets of Borel sets that have Lebesgue measure zero). All k-dimensional rectangles have Lebesgue measure given by the usual formulas for volume of a rectangle, and this determines it uniquely.

- **Counting measure**. S is a countable collection of points, $\mathcal{F}$ the class of all subsets of S. $\mu(A)$ is the number of points in A.

- Mixtures. Mix of Lebesgue measures on subsets or subspaces of different dimensions. For example, we might put some weight on the point $(x, y) = (1, 0)$, some weight on $(x, y) = (0, 1)$, some weight on $\{(x, y)\} \mid x + y = 1$, and some weight on the rest of $\mathbb{R}^2$.



Integrals

The Lebesgue integral of a function f over $E \subset S$ with respect to the measure μ is written as

$$\int_E f(\omega)\mu(d\omega) .$$

We will not write out the technical definition of the Lebesgue integral. For the cases we are considering, where μ is usually Lebesgue measure or some mixture of Lebesgue measures on lower-dimensional sets, the Lebesgue integral is what you would expect from undergrad calculus.

Densities

- p an integrable function on S
- Define a measure ν on S by

$$\nu(A) = \int_A p(\omega)\mu(d\omega)\,.$$

If $\int_S p\, d\mu = 1$ (note the alternate notation), ν is a probability measure and p is its **density** with respect to the measure μ. p is also called the probability density function or the pdf.

- Most common case: μ Lebesgue measure on $\mathbb{R}^n$.

- Mixed measures: There may not be a unique way to define μ. In that case the same ν may correspond to different p's depending on how the base measure μ is specified.

- Example: A distribution for (x, y) that puts probability .5 on the line $x = y$ over $(0, 0)$ to $(1, 1)$, and probability .5 on the rest of the unit square.
  - Base measure Lebesgue measure on the unit square plus Lebesgue measure (corresponding to length) on the $x = y$ line. $p(x, y) = .5$ for $x \neq y$ and $p(x, y) = 1/\sqrt{8}$.
  - Base measure gives unit weight to the $x = y$ line (instead of weight $\sqrt{2}$, corresponding to the line's length). $p = .5$ along $x = y$.