ANSWER TO FIRST PROBLEM SET

With the lump-sum tax on the young instead of the old, the first-order conditions remain unchanged from those in the model from class, since only the constant terms in agents' constraints are changed. So we end up as before with

$$\begin{aligned} \frac{RP_t}{P_{t+1}} &= \frac{C_{2,t+1}}{C_{1t}} & \text{if } B_t > 0 \\ \theta &= \frac{C_{2,t+1}}{C_{1t}} & \text{if } S_t > 0\,. \end{aligned}$$

The constraints have changed, though, to

$$\begin{aligned} C_{1t} + S_t + \frac{B_t}{P_t} &= 1 - \tau \\ C_{2,t+1} &= \frac{RB_t}{P_{t+1}} + \theta S_t\,. \end{aligned}$$

Using the problem's assumption that $S_t \equiv 0$, only the first of the two FOC equations above applies and S_t drops out of the two constraints. Taking the ratio of the two constraints gives us

$$\frac{C_{2,t+1}}{C_{1t}} = \frac{\frac{R}{P_{t+1}} B_t}{C_{1t}}\,. \qquad (*)$$

But

$$\frac{R}{P_{t+1}} B_t = \frac{RP_t}{P_{t+1}} \frac{B_t}{P_t} = \frac{C_{2,t+1}}{C_{1t}}(1 - \tau - C_{1t})\,, \qquad (\dagger)$$

where this last equality has used the first-period budget constraint to replace B_t/P_t with $1 - C_{1t} - \tau$. Using $(*)$ and $(\dagger)$ together we can arrive at

$$1 - \tau - C_{1t} = C_{1t}\,,$$

and therefore $C_{1t} = (1 - \tau)/2$.

Because there is no storage, $C_{1t} + C_{2t} = 1$ in every period, so $C_{2t} = (1 + \tau)/2$.

With the tax on the old, it was pointed out in class that as $\tau \to \infty$, $C_{1t} \to 0$, though proving this for the complicated expression for steady state C_{1t} in that model requires either a numerical exercise with a spreadsheet or a l'Hôpital's rule argument using calculus. So any C_{1t} between $\frac{1}{2}$ (the limiting value as $\tau \to 0$ for both the tax on the old and the tax on the young) and 0 is a possible steady state value, both for the tax on the old and the tax on the young. Since in either case

$C_{2t} = 1 - C_{1t}$, every allocation obtainable with one tax is obtainabale by the other. It can be shown that the required value of τ for a given allocation is always higher with the tax on the old, though you weren't asked to show that.

Some students (maybe most) thought at least initially that τ could never go above one. That's true for the tax on the young — all they have is their unit endowment, and taxes can't extract more than that. For the old, though, second period resources are the value of saved bonds. Since as τ increases, the return on bonds increases, the bonds held by the old end up exceeding one in value, indeed becoming arbitrarily large, as τ increases. So if it is a tax on the old, $\tau > 1$ is quite possible. It just taxes away the value of the bonds, so that what's left of the old's wealth has purchasing power less than one.

Some students were misled by observing that the expression in the notes for the approximate value of C_{1t} when the tax is on the old, $(1 - \tau)/2$, is also the exact solution when the tax is on the young. But this only shows that the solutions are approximately the same for small values of τ. As τ increases, the solutions for a given level of τ diverge.

The welfare of the typical generation is

$$\log C_{1t} + \log C_{2,t+1} = \log C_{1t} + \log(1 - C_{1,t+1}) \,.$$

Since C_{1t} is constant, this is

$$\log\big(C_{1t}(1 - C_{1t})\big) = \log\left(\frac{(1-\tau)(1+\tau)}{4}\right) = \log\left(\frac{1-\tau^2}{4}\right) \,.$$

Obviously, this is decreasing in τ, though at a very slow rate for τ near zero.

For the initial old, though, consumption is just savings of the initial young, which in this model is $(1 + \tau)/2$. Their welfare therefore increases with τ, and the effect of τ on their welfare is "first-order". That is, at $\tau = 0$, the derivative of welfare of the typical generation with respect to τ is zero, while the derivative of the welfare of the initial old with respect to τ is one. Even though there are infinitely many generations to benefit from low taxes and only one that benefits from the high taxes, the benefits to the future generations are tiny for small τ, and the offsetting losses to the current old are not so tiny. For example, a tax of .05 on the young reduces the welfare of current and future generations by about .0025, while it increases the welfare of the current old by about .05, a 40 times greater effect.

Finally, note that the solution to this model was neater than for the one with the tax on the old. There was no need to solve a quadratic equation. Also, we arrived at a unique solution for C_{1t} without having to bring in the steady-state assumption $C_{1t} = C_{1,t+1}$, so no separate argument that any other initial value of C_{1t} would lead to unsustainable dynamics was required.