EXERCISE ON NET WORTH AND LENDING RATES, DUE TUESDAY, 10/25

This exercise has you work through a simplified version of the model in Zha (2001, Section 2, "A Simple Case"). You should read that section of that article. You need not try to follow the mathematics of the more complicated dynamic model in the remainder of the paper, but you should read the discussion of its results, describing the capital accumulation and distributional effects of the bankruptcy exemption level.

Consider a firm that has the opportunity to invest in a project with a random rate of return A. The return is either A_u or A_ℓ, with each having probability .5. The project requires an investment of 1.0 if it is undertaken at all. The firm has wealth W initially. If $W > 1$, the firm can fund the project entirely out of its own wealth. If $W < 1$, the firm must borrow from the bank in the amount $1-W$ to fund the project. The bank charges a gross interest rate R^*, so that the loan contract requires payment of R^*B if the loan is in the amount B. If the investment is undertaken and $B > 0$, the loan contract specifies that if $A < R^*B$, the bank takes the entire return A and the firm is left with nothing. Because the risk of bankruptcy (i.e. of $R^*B < A$) depends on W, the bank's interest rate R^* will depend on W. There is a given market deposit rate R. The firm has the option of making no investment, taking out no loan, depositing its wealth W in the bank, and receiving a return RW. The bank requires that the expected rate of return on any loan it makes, that is, the expected value of $\min(R^*B,A)/B$, must be R. (We think of the bank as in a competitive banking sector. It must make an expected return of at least R to survive, and competition will force it not to demand an expected return higher than R.) We can represent these conditions formally as follows:

$$B = 1-W \text{ if } B > 0 \tag{1}$$

$$E\left[\frac{\min(R^*B,A)}{B}\right] = R \qquad \text{Bank return requirement} \tag{2}$$

$$E\left[\frac{\max(A-R^*B,0)}{W}\right] \geq R \qquad \text{firm return requirement} \tag{3}$$

(a) Suppose $R = 1.01$, $A_u = 1.1$, and $A_\ell = .96$. Plot $R^*(W)$, the bank's interest rate as a function of firm initial wealth W. Note that initial wealth could be negative Are there levels of wealth at which the bank will refuse to lend at any rate that the firm finds acceptable? Will the firm always want to borrow at the bank's rate $R^*(W)$ so long as $W < 1$?

If W is large enough so that there is no default, no matter what the value of A, then the bank's return is riskless and it will set $R^* = R = 1.01$. To find this level of wealth, we determine at what point $A_\ell \geq R(1-W)$. I.e., we solve .

$$.96 = 1.01(1-W)\,, \text{ finding } \quad W = \frac{.05}{1.01} = .0495\,.$$

For wealth levels below this, but not too low, there will be a 50% probability of bankruptcy — i.e. bankruptcy whenever $A = A_\ell$. At such wealth levels, the expected return condition for the bankers requires

$$\frac{\frac{1}{2}R^*(1-W)+\frac{1}{2}.96}{1-W} = R = 1.01\,.$$

Solving this gives us

$$R^* = 2.02 - \frac{.96}{1-W}\,.$$

Finally, we need to check at what point bankruptcy becomes a certainty. That is, solve

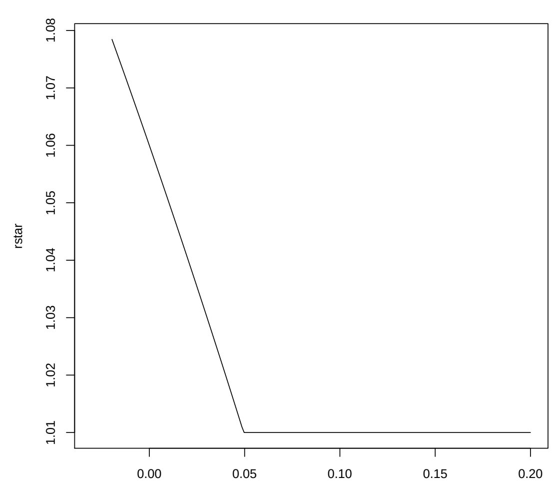
$$1.1 = R^*(1-W) = \left(2.02 - \frac{.96}{1-W}\right)(1-W)\,,\ \text{finding}\quad W = \frac{-.04}{2.02} = -.0198\,.$$

So, for $W > 1$, there is no borrowing and the firm invests one unit of wealth in the project, the rest in the deposits paying R. For $.0495 < W < 1$ the firm borrows, never goes bankrupt, and pays the rate $R^* = R$. For $-.0198 < W < .0495$, the firm borrows and pays a rate $R^* > 1.01$ when $A = 1.1$, goes bankrupt and ends up with 0 when $A = .96$. We still need to verify, though, that in this range with .5 probability of bankruptcy the firm is doing better by borrowing than putting the money in the bank. The problem does not make it explicit that firms can't get a "return" on W of R if R is negative. If they could, that would mean they can always borrow at R, which makes no sense. So if $W < 0$, the firm either invests, supported by a loan, or is stuck with its negative wealth. This means that it is worthwhile for the firm to borrow, when $W < 0$, even in the case where it goes bankrupt for sure, since that leave it with zero, rather than negative, wealth. The bank will be willing to lend in this certain-bankruptcy case, so long as $E[A]/(1-W) \geq R$, i.e. $1.03 \geq 1.01(1-W)$, which means $W > -.0198$. In other words, the bank will not lend to negative-wealth firms unless the wealth is at or above the same critical value that determines whether bankruptcy is certain. So there is no lending to firms that are certain to go bankrupt. For the $-.0198 < W < .0495$ range, what we need to check is

$$R \leq .5\frac{1.1 - R^*(1-W)}{W}$$

Solving this (by substituting out R^*) just leads to $0 \leq .02$, an identically true inequality. That is, the firm always wants to invest, no matter what level its W is at. Another way to see this is to observe that, because the firm and the bank split the returns of the investment, if one of them has an expected rate of return over 1.03 (the expected return to the investment as a whole), the other must have a return below 1.03, and vice versa. Since the bank is always setting R^* so its expected return is 1.01, The firm always gets an expected return over 1.03, whenever it borrows.

So the plot of R^* as a function of W looks like this, over the range where borrowing takes place, but truncating the large-W values for clarity:



(b) What if $R = 1.05$ rather than $R = 1.01$?

Now R exceeds the expected return on the investment as a whole. The bank is going to insist on getting this expected return, meaning the firm will be left with an expected return below 1.03. But the alternative of investing wealth at the rate $R > 1.03$ is available. So a firm with positive wealth never borrows. As we noted above, firms with negative wealth can't access the rate of return R, and they will be willing to borrow if the bank will lend, no matter what the rate of return, so long as bankruptcy leaves them at zero wealth. But even with bankruptcy certain, the bank's expected return is just $1.03/(1-W)$, and with $W \leq 0$, this is necessarily less than the critical value 1.05. So for positive W, firms are unwilling to borrow, and for negative W the bank is unwilling to lend. No borrowing or lending at all takes place with R at this level.

References

ZHA, T. (2001): "Bankruptcy Law, Capital Allocation, and Aggregate Effects: A Dynamic Heterogeneous Agent Model with Incomplete Markets," *Annals of Economics and Finance*, 2, 379–400.