ANSWERS FOR EXERCISES A AND B

Exercise A: Primary surplus is τ, now and forever. Required expected real return is ρ. Primary surplus will support current debt forever with no default risk (i.e. with $R_t \equiv \rho$):

$$b_t = \bar{b} = R_{t-1}b_{t-1} - \tau = \rho b_{t-1} - \tau = \rho\bar{b} - \tau$$

Now suppose that bondbuyers become nervous, think there might be a default, so require an interest rate $R > \rho$ in order to hold debt. If $b_t = \bar{b}$, then $b_{t+1} = R\bar{b} - \tau > \rho\bar{b} - \tau = \bar{b}$.

Prove: So long as $b_t > \bar{b}$ and $R > \rho$, $b_{t+1}/b_t > 1$ and b_{t+1}/b_t increases as b_t increases.

This is a case with real debt, so there is no P_t in the budget constraint, and it can be written as

$$b_t = Rb_{t-1} - \tau \,.$$

We will divide the constraint by b_{t-1}, then use the fact that $R > \rho$, then use the fact that $b_{t-1} > \bar{b}$.

$$\frac{b_t}{b_{t-1}} = R - \frac{\tau}{b_{t-1}} > \rho - \frac{\tau}{b_{t-1}} > \rho - \frac{\tau}{\bar{b}} = \rho - \frac{\tau}{\frac{\tau}{\rho-1}} = 1.$$

Furthermore, we can see from the first equality above that b_t/b_{t-1} is increasing in b_{t-1}.

Exercise B: Suppose τ_0 is 10, τ_1 is 8, the probability of switching from τ_o to τ_1 permanently is .1 every period, the amount of nominal debt initially is B_0, the real gross rate of return is ρ, and the nominal gross interest rate is R. Find the inflation or deflation every period before the drop, and the inflation rate at the date of the drop. Assume that after the drop everyone is certain that τ_1 will prevail forever.

The budget constraint is

$$b_t = R\frac{P_{t-1}}{P_t}b_{t-1} - \tau_t \,.$$

Since there is a real investment alternative delivering a gross return ρ, we assume that investors insist that the expected yield on government debt is also ρ, so

$$E_{t-1}\left[\frac{RP_{t-1}}{P_t}\right] = \rho \,.$$

Then before the switch we have

$$E_{t-1}b_t = \rho b_{t-1} - E_{t-1}\tau_t = \rho b_{t-1} - .9\tau_0 - .1\tau_1 = \rho b_{t-1} - 9.8 \,.$$

So long as b_t can't explode exponentially faster than ρ^t, we can solve this forward to get a before-the-switch value of real debt:

$$b_0 = \frac{9.8}{\rho - 1} \,.$$

After the switch, with τ fixed at $\tau_1 = 8$, solving forward gives us

$$b_1 = \frac{8}{\rho - 1}$$

At the date of the switch, we will have

$$b_t = b_1 = R\frac{P_{t-1}}{P_t} b_0 - \tau_1 \,.$$

Since we have already found formulas for b_0 and b_1, we can solve this for the value of $\Pi_1 = P_t / P_{t-1}$ at the time of the switch:

$$\Pi_1 = \frac{b_0 R}{b_1 + \tau_1} = \frac{\frac{9.8R}{\rho-1}}{\frac{8}{\rho-1} + 8} = \frac{9.8R}{8\rho} \,.$$

When the switch has not occurred, we will instead have

$$b_t = b_0 = b_0 \frac{R}{\Pi_0} - \tau_0 \,,$$

which we can solve to find

$$\Pi_0 = \frac{9.8}{10 - .2/\rho} \,.$$

When $\rho = R$, the gross inflation at the time of the switch is $9.8/8 = 1.225$, and before the switch is

$$\frac{9.8}{10 - .2/\rho} < 1 \,,$$

so there is slight deflation at every date before the switch, then a substantial inflation at the date of the switch.