First Order Conditions for Stochastic Problems: Examples

I. Permanent income model

A. $\max_{\{C(t),A(t)\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t U\left(C(t)\right)\right]$

B. $A(t) \le (1+r_{t-1})A(t-1) - C(t) + Y(t),\ t = 1,\ldots,\infty$,

1. *or* $W(t) \le (1+r_{t-1})\left(W(t-1) - C(t-1)\right) + Y(t)$

   a) (follows from $W(t) = (1+r_{t-1})A(t-1) + Y(t)$)

C. FOC's

1. ∂C: $U'(C(t)) - \lambda(t) = 0$
2. ∂A: $\lambda(t) = \beta \cdot (1+r_t)E_t\left[\lambda(t+1)\right]$

D. Transversality

1. $E\left[\sum_{t=0}^{T} \lambda_t \beta^t \left\{dA(t) + dC(t) - dA(t-1)(1+r_{t-1})\right\}\right] = E\left[\sum_{t=0}^{T} \lambda_t \beta^t dC(t)\right] + E\left[\lambda_T \beta^T dA(T)\right]$

   converges, as a linear function of {dC, dA}, as $T\to\infty$, and the same holds true for $E\left[\sum_{t=0}^{T} \beta^t U_t' dC_t\right]$.

   a) What is usually called the transversality condition is one component of the condition we have stated here, checked at *dA*=*A*: $\lim_{T\to\infty} E\left[\beta^T \lambda_T A(T)\right] = 0$. If the interest rate *r* is constant, it is feasible to set $C(t) = 0$, all *t* and thereby make *A* grow as $(1+r)^t$. Thus even if $A(t)$ remains bounded as $t \to \infty$ at the optimum, application of the separating hyperplanes argument requires that in addition $\lim_{T\to\infty} E\left[\beta^T \lambda_T (1+r)^T\right] = 0$. If $\beta \cdot (1+r) < 1$ this will be true whenever $E\lambda_T = EU_T'$ remains bounded on the optimum path. If $\beta \cdot (1+r) \ge 1$ it can be true only if $E\lambda_T$ tends rapidly enough to zero.

   b) Note that to actually check the condition, we need to know the nature of the solution for optimal *C* and *A*, which will in turn depend on the properties of the exogenous random terms *r* and *Y*.

II. Optimal Growth

A. $\max\limits_{\{K(t),I(t),C(t),L(t)\}} E\left[\sum_{t=0}^{\infty} \beta^t U\big(C(t), 1-L(t)\big)\right]$

B. subject to

1. λ : $C(t)+I(t)=A(t)f(K(t-1),L(t))$

2. μ $K(t)=(1-\delta)K(t-1)+I(t)$

C. FOC's

1. ∂C: $D_1U_t=\lambda(t)$

2. ∂L: $D_2U_t=\lambda(t)A(t)D_L f_t$

3. ∂I: $\lambda(t)=\mu(t)$

4. ∂K: $\mu(t)=\beta E_t\big[\lambda(t+1)A(t+1)D_K f_{t+1}+(1-\delta)\mu(t+1)\big]$

D. Transversality

1. $E\left[\sum_{t=0}^{T}\beta^t\lambda_t\big(dC(t)+dI(t)-A(t)\{D_K f_t dK(t-1)+D_L f_t dL(t)\}\big)\right]+$

$E\left[\sum_{t=0}^{T}\beta^t\mu_t\big(dK(t)-dI(t)-(1-\delta)dK(t-1)\big)\right]=$

$E\left[\sum_{t=0}^{T}\beta^t\lambda_t\big(dC(t)+D_L f_t dL_t\big)\right]+E\big[\beta^T\mu_T dK(T)\big]$ converges. In particular, $\lim\limits_{T\to\infty}E\big[\beta^T\mu_T K(T)\big]=0$. Also, $E\left[\sum_{t=0}^{T}\beta^t\big(D_1U_t dC(t)-D_2U_t dL(t)\big)\right]$ converges.

III. Exercise

A. Find the FOC's for the permanent income model with *W*'s replacing *A*'s. Show the relationship between the Lagrange multiplier for this version of the problem and that shown above for the *A* version.

B. Show that if r_t varies randomly, independently across time, and if $E\big[(1+r_t)\beta\big]=1$, this is *not* enough to imply that marginal utility of consumption is even approximately a martingale, so long as the variation in *r* is large.

C. Assume the marginal utility of leisure is zero, so that optimal *L* is identically 1 and it drops out of the decision problem in the growth model. Suppose $U_t=\log C_t$, $\delta=1$ and $f_t=K_{t-1}^{\alpha}$. Show that a constant $C(t)/K(t)$ satisfies the

FOC's, including transversality. Assume $0 < a < 1$. Assume the exogenous stochastic process *A* is bounded away from zero and infinity.

D. For the same case as in part C above, show that no other feasible solution to the Euler equations (the FOC's other than transversality) satisfies transversality.