# Macroeconomic Theory Econ 511b
# Answers to Problem Set #1

January 30, 1998

**Question A:**
The problem is:

$$\max_{\{C_t, W_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t U(C_t)\right]$$

$$s.t. \qquad W_t = (1 + r_{t-1})(W_{t-1} - C_{t-1}) + Y_t \qquad (\mu_t)$$

The first order conditions are:
$\frac{\partial}{\partial C_t}$ :

$$U'(C_t) = \beta E_t [\mu_{t+1}(1 + r_t)]$$

and:
$\frac{\partial}{\partial W_t}$:

$$\mu_t = \beta E_t [\mu_{t+1}(1 + r_t)]$$

If we combine these we get:

$$U'(C_t) = \mu_t$$

But this just the first order condition with respect to $C_t$ from the problem discussed in class. Thus $\mu_t \equiv \lambda_t$.

**Question B:**
The solution to this problem is simply to recognize that according to the Euler equation

$$U'(C_t) = (1 + r_t)\beta E_t [U'(C_{t+1})]$$

the marginal utility of consumption is only a martingale if $(1 + r_t)\beta \equiv 1$. With a small enough variation in $r_t$, this discrepancy is negligible.

**Question C:**

The problem is:

$$\max_{\{C_t,K_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t \log C_t\right]$$

$$C_t + K_t = A_t K_{t-1}^{\alpha}$$

First order conditions are:

$\frac{\partial}{\partial C_t}$ :

$$\frac{1}{C_t} = \lambda_t$$

and

$\frac{\partial}{\partial K_t}$ :

$$\lambda_t = \alpha\beta E_t\left[\frac{A_{t+1}K_t^{\alpha-1}}{C_{t+1}}\right]$$

or:

$$\frac{1}{C_t} = \alpha\beta E_t\left[\frac{A_{t+1}K_t^{\alpha-1}}{C_{t+1}}\right]$$

Suppose now that $K_t/C_t = \kappa$. Plug this into the Euler equation and after some manipulation you find $\kappa = \alpha\beta/(1-\alpha\beta)$. Consequently, the decision rules are:

$$\begin{aligned} K_t &= \alpha\beta A_t K_{t-1}^{\alpha} \\ C_t &= (1-\alpha\beta)A_t K_{t-1}^{\alpha} \end{aligned}$$

A constant consumption-capital ratio thus satisfies the first order conditions, but does it also satisfy transversality? We need to check whether the following expression converges as $T \to \infty$:

$$E\left[\sum_{t=0}^{T}\beta^t \lambda_t dC_t\right] + E\left[\beta^T \lambda_T dK_T\right]$$

We know that $\lambda_T = 1/C_T$ and that $K_T = \kappa C_T$. Evaluating the two expressions at the optimal choices $dC_t = C_t$ and $dK_T = K_T$ gives the following:

$$\begin{aligned} E\left[\sum_{t=0}^{\infty}\beta^t \frac{1}{C_t} C_t\right] &= 1/(1-\beta) \\ \lim_{T\to\infty} E\left[\beta^T \frac{1}{C_T}\kappa C_T\right] &= 0 \end{aligned}$$

**Question D:**

The FOC's imply:

$$\frac{1}{C_t} = \alpha\beta E_t\left[\frac{A_{t+1}K_t^{\alpha-1}}{C_{t+1}}\right]$$

and in part C) it should already have been verified that $\kappa_t = K_t/C_t$ set at $\overline{\kappa} = \alpha\beta/(1-\alpha\beta)$ provides a solution. Once we have fixed this ratio, $K$ and $C$ are determined from time $t = 0$ onward. Initial total output is determined by $K_{t-1}$ and $A_t$, and the ratio determines how we split it between $C_t$ and $K_t$. Any other solution to the FOC's must still satisfy the Euler equation. Using the technology constraint, we can rewrite the equation as:

$$\kappa_t = \alpha\beta E_t\left[\kappa_{t+1}\right] + \alpha\beta.$$

This is an explosive difference equation in $E_t\left[\kappa_{t+s}\right]$ as a function of $s$, so long as $\alpha\beta < 1$, which it is. Therefore, if $\kappa_0 < \overline{\kappa}$, $E_0\left[\kappa_t\right]$ becomes negative eventually, implying a non-zero probability of observing a negative $K$, which is impossible. If $\kappa_0 > \overline{\kappa}$, $\kappa_t$ explodes upward at the rate $(\alpha\beta)^{-1}$. But one aspect of transversality is the requirement, given in the notes, that:

$$\lim_{T\to\infty} E\left[\beta^T\mu_T K_T\right] = \lim_{T\to\infty} E\left[\beta^T\frac{K_T}{C_T}\right] = \lim_{T\to\infty} E\left[\beta^T\kappa_T\right] = 0.$$

Obviously the last equality is true if $\kappa$ is constant, and false if $\kappa_t$ behaves like $(\alpha\beta)^{-T}$. So transversality is violated by all solutions to the Euler equation other than the solution that makes $\kappa$ constant.