# Linear Rational Expectations Model Exercise Answers

(i) - (iii):

The FOC's in the general case are

$\partial C$: $$1 - C_t = \lambda_t \qquad [1]$$

$\partial A$: $$\lambda_t = \beta E_t\left[\lambda_{t+1}\cdot(1+r)\right] + \mu_t \qquad [2]$$

with $\mu_t = 0$ when (3) is not binding, and $\lambda_t = 0$ when (2) is not binding. (Equations in this answer sheet are in square brackets, while those in the original problem set are in ordinary parentheses.) To answer (ii) first, just set $\lambda_t \equiv 0$ in [1] and [2], to obtain

$$C_t = 1 \qquad [3]$$

$$\mu_t = 0\,. \qquad [4]$$

Because there is no dynamic constraint active, there is no conventional transversality condition in this version of the problem.

The equation system in *C*, *A*, and $\mu$ when [3] and [4] are combined with the binding constraint (3), is

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}C_t\\ A_t\\ \mu_t\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}. \qquad [5]$$

If we introduce a variable $X_t = Y_{t-1}$ to cast the full system into first-order form, the equation describing the evolution of the exogenous random income variable $Y_t$ becomes the two-equation system

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}Y_t\\ X_t\end{bmatrix} = \begin{bmatrix}1.5 & -.75\\ 1 & 0\end{bmatrix}\begin{bmatrix}Y_{t-1}\\ X_{t-1}\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} z_t\ . \qquad [6]$$

To form the full 5x5 system in *C*, *A*, $\mu$, *Y*, and *X*, we have to combine [5] and [6]. Since each involves a separate list of variables, the $\Gamma$ matrices of coefficients will be block diagonal. In particular, we will have

$$\Gamma_0 = I,\ \Gamma_1 = \begin{bmatrix}0 & 0\\ 0 & \begin{bmatrix}1.5 & -.75\\ 1 & 0\end{bmatrix}\end{bmatrix} \qquad [7]$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \Pi = 0, C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ .25 \\ 0 \end{bmatrix}. \qquad [8]$$

For (i) the system is more complicated. The set of equations we will use, in which $\lambda$ takes the place of $\mu$ from the system above, is obtained by setting $\mu$ to zero in [1] and [2] and stacking up the results with the active constraint and the exogenous variable dynamics, to arrive at

$$\Gamma_0 = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \beta \cdot (1+r) & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1+r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.75 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad [9]$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \Pi = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} -1 \\ 0 \\ 0 \\ .25 \\ 0 \end{bmatrix}. \qquad [10]$$

Here the variables are ordered $C, A, \lambda, Y, X$.

iv):

We don't really need the apparatus of matrix decompositions to check the solution for the case where (3) always binds. This implies $C$ is identically one, which is the satiation level of consumption. No higher value of the objective function is obtainable than what we get with $C$ identically one (which is $1/(2 \cdot (1-\beta))$. The only question is whether this is feasible. Recall that in this solution we *assume* that (2) never binds. Is this possible? With $A$ held at zero by the binding constraint, (2) simply asserts $C_t \le Y_t$. With $C_t$ always equal to one, this means that $Y$ must always equal or exceed one. Under these circumstances, when current income always suffices to support the satiation level of consumption, it is clearly feasible and optimal to accumulate no capital and to consume at the satiation level at all times. But is a level of $Y$ that always equals or exceeds one consistent with the constraint on exogenous variables given by (4)? Just barely. Taking unconditional expectations of all terms in that equation allows us to conclude that $EY = 1$. If $Y \ge 1$ with probability one, while also $EY = 1$, then necessarily $Y = 1$ with probability one. Thus for this solution to be viable and consistent with all the equations of the system, it would have to be true that there is no random disturbance in (4), or equivalently that $z_t = 0$ with probability one for all *t*.

For the other case, where (2) but not (3) binds, we do need to carry out matrix decompositions. For a general 5x5 matrix this would be impractical as a hand calculation, but the matrices in [9] are block triangular, with a 3x3 block in the upper left and a 2x2 in the lower right. This makes all the hard calculations 2x2 or 3x3, which is quite feasible by hand. The first task is to invert $\Gamma_0$. The inverse of a block triangular matrix is block triangular with the same structure and has the inverses of its diagonal submatrices on the diagonal. Once we have computed these, the upper right block of the inverse is found directly from solving the equations in the upper right block of $\Gamma_0^{-1}\Gamma_0 = I$.

To be specific, the lower right block of the inverse is just the identity. The inverse of the upper left block is found by straightforward calculation to be

$$\begin{bmatrix} -1 & -\beta^{-1}\cdot(1+r)^{-1} & 0 \\ 1 & \beta^{-1}\cdot(1+r)^{-1} & 1 \\ 0 & \beta^{-1}\cdot(1+r)^{-1} & 0 \end{bmatrix}. \qquad [11]$$

If we let $H$ stand for the upper right corner of $\Gamma_0^{-1}$, then the condition we solve to get $H$ is

$$\begin{bmatrix} -1 & -\beta^{-1}\cdot(1+r)^{-1} & 0 \\ 1 & \beta^{-1}\cdot(1+r)^{-1} & 1 \\ 0 & \beta^{-1}\cdot(1+r)^{-1} & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} = -H\cdot I, \qquad [12]$$

which reduces to

$$H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \qquad [13]$$

The matrix whose roots we need to analyze is $\Gamma_0^{-1}\Gamma_1$, or

$$\begin{bmatrix} -1 & -\beta^{-1}\cdot(1+r)^{-1} & 0 & 0 & 0 \\ 1 & \beta^{-1}\cdot(1+r)^{-1} & 1 & 1 & 0 \\ 0 & \beta^{-1}\cdot(1+r)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1+r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.75 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \qquad [14]$$

This matrix multiplication is easy because every column of the right-hand matrix except the fourth has only a single element, so that every column of the product except the fourth is just a scalar multiple of a column of the left-hand matrix. The result is

$$\begin{bmatrix} 0 & 0 & -\beta^{-1}\cdot(1+r)^{-1} & 0 & 0 \\ 0 & 1+r & \beta^{-1}\cdot(1+r)^{-1} & 1.5 & -.75 \\ 0 & 0 & \beta^{-1}\cdot(1+r)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.75 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad [15]$$

The characteristic polynomial of this matrix is easily found as

$$\left(\beta^{-1}\cdot(1+r)^{-1}-\lambda\right)\cdot(1+r-\lambda)\cdot(-\lambda)\left(\lambda^2-1.5\lambda+.75\right). \qquad [16]$$

The roots of the quadratic factor in this expression are complex and equal to $.866e^{\pm i\pi/3}$. Thus they give rise to a component of the solution that decays in magnitude as $.866^t$ (which has a half life of 4.8 years) and oscillates with a period of 6 years. So long as the interest rate is positive, there is at least one unstable root in the system, given by 1+*r*. The other non-zero root, $\beta^{-1}\cdot(1+r)^{-1}$, may or may not exceed one in absolute value. The conventional transversality condition is

$$\beta^t E[\lambda_t A_t] \to 0 \ . \qquad [17]$$

We know from [2] that $E[\lambda_t]$ grows at the rate $\beta^{-t}\cdot(1+r)^{-t}$. Therefore we expect that transversality will rule out solutions with components that grow as fast as $(1+r)^t$. Thus we will impose as a stability condition that the component of the solution corresponding to the 1+*r* root be suppressed. The other non-zero root, $\beta^{-1}\cdot(1+r)^{-1}$, may or may not equal or exceed 1+*r*, according to whether or not $\beta^{-1} \le (1+r^2)$. So we have to consider some cases.

First, suppose 1+*r* is strictly the largest root. To find a left eigenvector of [15] of the form $[a\;1\;b\;c\;d]$ corresponding to the eigenvalue $\rho$, we solve the equation

$$[a\;1\;b\;c\;d]\cdot\begin{bmatrix} 0 & 0 & -\beta^{-1}\cdot(1+r)^{-1} & 0 & 0 \\ 0 & 1+r & \beta^{-1}\cdot(1+r)^{-1} & 1.5 & -.75 \\ 0 & 0 & \beta^{-1}\cdot(1+r)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.75 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \rho\cdot[a\;1\;b\;c\;d] \ .$$

(Why not [1 *a b c d*] instead of [*a* 1 *b c d* ]? Because it turns out the first element has to be zero, so normalizing on the first element of the vector only works for the $\rho = 0$ root.) For $\rho = 1+r$, this becomes the system

$$
\begin{aligned}
0 &= (1+r)\cdot a \\
1+r &= 1+r \\
(1+b-a)\beta^{-1}(1+r)^{-1} &= (1+r)\cdot b \\
1.5(1+c)+d &= (1+r)\cdot c \\
-.75(1+c) &= (1+r)d
\end{aligned}
\quad . \qquad [18]
$$

Clearly $a=0$ and $b = 1\Big/\left((1+r)^2\beta - 1\right)$. The coefficients $c$ and $d$ can then be solved for from the last two equations of [18] and are found to be

$$
c = \frac{3+6r}{4r^2+2r+1}\,,\quad d = \frac{-3\cdot(1+r)}{4r^2+2r+1}\ . \qquad [19]
$$

The stability condition has the form, then,

$$
-\lambda_t = \left((1+r)^2\beta - 1\right)\left(A_t + cY_t + dY_{t-1} - .75\right)\ . \qquad [20]
$$

The existence condition is that the column space of

$$
[0\;1\;b\;c\;d]\Gamma_0^{-1}\Psi = 1+c \qquad [21]
$$

be spanned by that of

$$
[0\;1\;b\;c\;d]\Gamma_0^{-1}\Pi = \beta^{-1}\cdot(1+r)^{-1}(1+b). \qquad [22]
$$

This will always be satisfied, because as long as $\beta$ and 1+$r$ are positive, $b > -1$. (Of course in this simple example $b$ and $c$ are scalars, so this "spanning" condition just requires that the right-hand side of [22] be non-zero if the right-hand side of [21] is. From [19] it is easy to check that $1+c \neq 0$, so that the stability condition fixes the value of $\eta$ and thereby guarantees there is no possibility of non-uniqueness.

So our conclusion is that there is in this case a unique solution to all the FOC's including conventional transversality.

When $(1+r)^2\beta \leq 1$, then 1+$r$ is no longer the largest root, and both roots, 1+$r$ and $(1+r)^{-1}\beta^{-1}$, would generate violations of transversality if present. We expect generally that when there is only one endogenous error, no more than one stability condition can be imposed, though there will be exceptions. The second stability condition has coefficients found from

$$
\begin{aligned}
0 &= \beta^{-1}\cdot(1+r)^{-1}\cdot a \\
(1+r)^{-1}\beta^{-1} &= (1+r)^{-1}\beta^{-1} \\
(1+b-a)\beta^{-1}(1+r)^{-1} &= (1+r)^{-1}\beta^{-1}\cdot b \\
1.5(1+c)+d &= \beta^{-1}(1+r)^{-1}\cdot c \\
-.75(1+c) &= \beta^{-1}(1+r)^{-1}d
\end{aligned}
\quad . \qquad [23]
$$

This system's solution has $a = b = 0$, so it implies as a stability condition

$$A_t = -c_2 Y_t - d_2 Y_{t-1} - .75 + .25c\,. \qquad [24]$$

Checking for existence, we form

$$\begin{bmatrix} 0 & 1 & 1\Big/\left((1+r)^2\mathrm{b}-1\right) & c_1 & d_1 \\ 0 & 1 & 0 & c_2 & d_2 \end{bmatrix} \Gamma_0^{-1}\Psi = \begin{bmatrix} 1+c_1 \\ 1+c_2 \end{bmatrix}, \qquad [25]$$

where $c_j$, for example is the $c$ component of the eigenvector associated with the $j$'th root,

and

$$\begin{bmatrix} 0 & 1 & 1\Big/\left((1+r)^2\mathrm{b}-1\right) & c_1 & d_1 \\ 0 & 1 & 0 & c_2 & d_2 \end{bmatrix} \Gamma_0^{-1}\Pi = \begin{bmatrix} (1+r)\Big/\left((1+r)^2\mathrm{b}-1\right) \\ \mathrm{b}^{-1}(1+r)^{-1} \end{bmatrix}. \qquad [26]$$

As is usual in cases like this with more unstable roots than h's, we find here that the right-hand side of [25] is not in the space spanned by (i.e., in this 2x1 case, is not a scalar multiple of) the right-hand side of [26]. (The algebra is a little tedious. Note that in this case the upper term on the right-hand side of [26] is negative, and the lower term positive. It is not too hard to show that both components of the right-hand side of [25] are positive. Thus no solution exists in this case.

Having done all this work, we must now note an unpleasant fact. Not only has the second case given us no equilibrium, but the first case, which seemed to produce a unique solution, has actually given us a spurious solution. In the second case, the problem is just that the true solution generally has (2) binding for a while, then (3) binding. That is, in this case, where discounting of the future is very heavy, optimal behavior involves deliberately consuming available wealth, until it is gone and (3) becomes binding. Thus the message that our linear system cannot produce a solution is correct.

The first case, though, seems to satisfy every condition for an optimum. It has a concave objective function, a convex constraint set, and satisfies all FOC's, including conventional transversality. However, this is a case where the conventional transversality condition is inadequate. It may be easiest to see that the proposed solution is not a solution by a direct argument. If $(1+r)^{-1}\mathrm{b}^{-1} > 1$ the solution will make $A$ explode at that rate, and to keep (3) non-binding it will have to explode in a positive direction. But then from the form of the stability condition, which makes $C$ depend positively on $A$, this implies $C$ also eventually explodes upward – exceeding 1, eventually. But we can certainly improve on any solution that implies $C$ ever exceeds one. To do so, at every date where $C$ exceeds one we set $C$=1, while keeping the $A$ time path unchanged. This of course makes (2) non-binding at these dates, but this is certainly feasible. Since $C$ is the satiation level of consumption, this altered policy improves utility in comparison with our apparent solution to the problem. If $(1+r)^{-1}\mathrm{b}^{-1} < 1$, the FOC [2] implies that $\mathrm{l}_t$ converges exponentially toward zero in expectation. If $C$ ever becomes greater than one, then we can improve on the solution as above just by switching to $C$=1 on the dates where the supposed solution says to have $C$>1. If not, then we have from [2] that $E_t C_{t+1} > C_t$, all $t$. This allows us to apply a useful result known as the ***martingale convergence theorem*****:**

If $X$ is a stochastic process satisfying $E\left[X_{t+1}\right] \geq X_t$, all $t$ (which makes it a *submartingale*), and if there is a random variable (or a constant) $Y$ such that $E|Y| < \infty$ and $X_t < Y$, all $t$, then $X$ converges with probability one.

Convergence of $X$ with probability one means that the probability of an $X$ sequence that fails to converge as an ordinary sequence of real numbers is zero.

The theorem clearly implies that, if $C$ remains always below one, it must converge to something. But the decision rule, together with the constraint, implies that $C$ always depends on the realization of $Y$ with a fixed coefficient. Hence the uncertainty about its value one step ahead is always the same, and it cannot possibly be converging to anything.

The true solution to the problem can be found only numerically. It will be well approximated by the decision rule we have derived for the linear system in the range where it implies $C_t$ considerably less than one. As the linear decision rule starts to imply $C$ close to one, the true rule starts to imply lower $C$ (hence higher saving and more rapid growth in $A$), and when $A$ gets large enough that interest on $A$ alone exceeds one, $C$ becomes exactly one.

What's wrong with transversality here? The problem is that the $A$ path at the solution is not representative of the full range of $A$'s in the feasible set. For the separating hyperplane theorem to apply, we need that $E\left[\lambda_t \beta^t A_t\right] \to 0$ for all feasible $A$ paths, not just for the $A$ path corresponding to the proposed solution. (Here we consider varying $A$ while holding the stochastic process for $\lambda$ fixed, because this condition is derived from the need to check that $E\left[\sum_t \lambda_t \left(D_{A_t} g \cdot dA_t + D_{A_{t+1}} g dA_{t+1}\right)\right]$ converges, as a function of the $dA$'s.) We know from [2] that $E\left[\lambda_t \beta^t\right]$ has a time path proportional to $\beta^{-t}(1+r)^{-t}$. It is feasible to choose an $A$ path that is deterministic and proportional to $(1+r)^t$. For such a path, clearly $E\left[\lambda_t \beta^t A_t\right]$ converges to a non-zero constant, violating the true transversality condition.