Linear Approximation to NeoClassical Growth

1. Linearizing the neoclassical stochastic growth model

Consider the model in which a representative agent maximizes

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$$

subject to

$$\begin{aligned}
&\left(\theta C_t^{\mu}+(2-\theta)I_t^{\mu}\right)^{1/\mu}=A_t(\alpha K_{t-1}^{\sigma}+(1-\alpha)L_t^{\sigma})^{1/\sigma}\\
&K_t=\delta K_{t-1}+I_t\\
&L_t=1
\end{aligned}\tag{1}$$

If we use λ and ν to denote the Lagrange multipliers for the two constraints, the first order conditions for an optimum (other than transversality) are

$$\begin{aligned}
&\frac{\partial}{\partial C}:\ \ C_t^{-\gamma}=\lambda_t\theta\cdot\left(\frac{Y_t}{C_t}\right)^{1-\mu}\\
&\frac{\partial}{\partial I}:\ \ \lambda_t\cdot(2-\theta)\left(\frac{Y_t}{I_t}\right)^{1-\mu}=\nu_t\\
&\frac{\partial}{\partial K}:\ \ \beta E_t\left[\lambda_{t+1}A_{t+1}^{\sigma}\alpha\cdot\left(\frac{Y_{t+1}}{K_t}\right)^{1-\sigma}\right]=\nu_t-\delta\beta E_t\left[\nu_{t+1}\right]\\
&Y_t=\left(\theta C_t^{\mu}+(2-\theta)I_t^{\mu}\right)^{1/\mu}
\end{aligned}\tag{2}$$

It simplifies notation and interpretation if we introduce special symbols for the derivative of "output" Y with respect to investment goods I and consumption C

$$\begin{aligned}
Q_t&=(2-\theta)\left(\frac{Y_t}{I_t}\right)^{1-\mu}\\
P_t&=\theta\left(\frac{Y_t}{C_t}\right)^{1-\mu}
\end{aligned}\tag{3}$$

and introduce another special symbol for the rate of return on capital

$$r_t = \frac{A_t^\sigma \alpha}{Q_t} \cdot \left(\frac{Y_t}{K_{t-1}} \right)^{1-\sigma} + \delta . \tag{4}$$

This allows rewriting the first three equations of (2) as

$$\begin{aligned} &\frac{\partial}{\partial C}: \;\; C_t^{-\gamma} = \lambda_t P_t \\ &\frac{\partial}{\partial I}: \;\; \lambda_t \cdot Q_t = \nu_t \\ &\frac{\partial}{\partial K}: \;\; \beta E_t\left[\lambda_{t+1} Q_{t+1} r_{t+1}\right] = \lambda_t Q_t \end{aligned} \quad , \tag{5}$$

which can in turn be solved to eliminate λ and ν and yield

$$E_t\left[\frac{C_{t+1}^{-\gamma} Q_{t+1} P_t r_{t+1}}{C_t^{-\gamma} Q_t P_{t+1}} \right] = \beta^{-1} \; . \tag{6}$$

Equation (6) equates the subjective discount factor to the expected rate of return on capital, corrected for the rate of inflation in capital goods prices in consumption goods units and for the intertemporal marginal rate of substitution of consumption goods. It is not hard to see from (6) that in deterministic steady state,

$$r = \beta^{-1}. \tag{7}$$

Of course in this model Q, P, and r, are not actually transaction prices, since there is a single type of representative agent and therefore no trade.

With (7) in hand we do not need an explicit solution for the steady state in order to linearize, but if the solution is needed, it can be found by converting (7) to the form

$$\frac{\alpha Y^{\mu-\sigma} I^{1-\mu}}{(2-\theta) K^{1-\sigma}} + \delta = \frac{\alpha (1-\delta)^{1-\mu}}{(2-\theta)} \left(\frac{Y}{K} \right)^{\mu-\sigma} + \delta = \beta^{-1}. \tag{8}$$

Equation (8) can be solved for steady state K explicitly, once the right hand side of (1) with L=1 is substituted for Y. To keep our system one in just three variables, we must rewrite (6) in terms of C, K, I alone, using the definitions of Q, P, and r:

$$E_t\left[\frac{C_{t+1}^{1-\gamma-\mu} I_t^{1-\mu}}{C_t^{1-\gamma-\mu} I_{t+1}^{1-\mu}} \cdot \left(\frac{A_{t+1} \alpha \cdot \left(\alpha + (1-\alpha) K_t^{-\sigma} \right)^{\frac{1-\sigma}{\sigma}}}{(2-\theta)\left(\theta \cdot (C_{t+1} / I_{t+1})^\mu + 2 - \theta \right)^{\frac{1-\mu}{\mu}}} + \delta \right) \right] = \beta^{-1} \tag{9}$$

It is convenient to linearize with respect to the natural logarithms of *C*, *K* and *I* rather than the levels of the variables. We will form a Taylor expansion in the logs of these variables about the steady state of the system given by the equations in (1) (with *L*=1 substituted out) together with (8). We will use lower case letters to refer to logs of variables and *dx* to refer, for any variable *X*, to $\log(X)-\log(\overline{X})$, where $\overline{X}$ is the steady state value of *X*. Then the linearized system is

$$\overline{P}\overline{C}dc_t+\overline{Q}\overline{I}di_t=(\overline{r}-\delta)\overline{Q}\overline{K}dk_{t-1} \tag{10}$$

$$dk_t=\delta dk_{t-1}+(1-\delta)di_t \tag{11}$$

$$\begin{aligned}&\left(1-\gamma-\mu-\theta\cdot(1-\mu)\frac{\overline{r}-\delta}{\overline{r}}\left(\frac{\overline{C}}{\overline{Y}}\right)^{\mu}\right)dc_{t+1}-\left(1-\mu-\theta\cdot(1-\mu)\frac{\overline{r}-\delta}{\overline{r}}\left(\frac{\overline{C}}{\overline{Y}}\right)^{\mu}\right)di_{t+1}\\&=(1-\gamma-\mu)dc_t-(1-\mu)di_t+\frac{\overline{r}-\delta}{\overline{r}}(1-\sigma)(1-\alpha)\overline{Y}^{-\sigma}dk_t+\frac{\overline{r}-\delta}{\overline{r}}da_{t+1}+\eta_{t+1}\end{aligned} \tag{12}$$

In (12), the η_{t+1} term is an endogenous expectational error and satisfies $E_t\eta_{t+1}=0$. We can see from (8) that steady state values of *K* , and hence (via the constraints in (1)) *C* and *I*, are all independent of the parameter γ. Thus we can see from (12) that if γ is made large while the other parameters remain fixed, the system has an equation that in the limit is approximately $E_t dc_{t+1}=dc_t$, apparently making the log of consumption a martingale. While this turns out to be correct, the argument we have just given is not tight, as making γ large might make the values of *K* along the solution path large, so that the fact that the coefficients on *K* are small might not imply that they have small influence on $E_t dc_{t+1}$.

To complete an analysis of the conditions under which the Hall model's conclusion that *c* is approximately a martingale is valid, we need to form the system's characteristic polynomial. Calling the five coefficients on choice variables in (12) v_i, $i=0,\ldots,4$, we can lag (12) once and combine it with (10) and (11) to obtain the following system in matrix notation:

$$\begin{bmatrix}PC & QI & 0\\ 0 & -1+\delta & 1\\ v_0 & v_1 & 0\end{bmatrix}\begin{bmatrix}dc_t\\ di_t\\ dk_t\end{bmatrix}=\begin{bmatrix}0 & 0 & (r-\delta)QK\\ 0 & 0 & \delta\\ v_2 & v_3 & v_4\end{bmatrix}\begin{bmatrix}dc_{t-1}\\ di_{t-1}\\ dk_{t-1}\end{bmatrix}+\begin{bmatrix}\overline{Y}\\ 0\\ \dfrac{r-\delta}{r}\end{bmatrix}da_t+\begin{bmatrix}0\\0\\1\end{bmatrix}\eta_t\ . \tag{13}$$

Note that because all the variables are taken as deviations from steady state, the linearized system contains no constant terms. The left-hand-side matrix is non-singular for all possible parameter values, because it can be shown that v_0 and v_1 must be of opposite sign (with $v_0<0$), so that the determinant $\Delta=QIv_0-PCv_1$ is always non-zero. It therefore has an inverse we can calculate as

$$\frac{1}{\Delta}\begin{bmatrix} -v_1 & 0 & QI \\ v_0 & 0 & -PC \\ (1-\delta)v_0 & QIv_0 - PCv_1 & -(1-\delta)PC \end{bmatrix}. \tag{14}$$

Multiplying the matrix in (14) times the right-hand-side square matrix in (13) yields

$$\frac{1}{\Delta}\begin{bmatrix} v_2 QI & v_3 QI & v_4 QI - v_1\cdot(r-\delta)QK \\ -v_2 PC & -v_3 PC & -v_4 PC + v_0\cdot(r-\delta)QK \\ -v_2 PC\cdot(1-\delta) & -v_3 PC\cdot(1-\delta) & v_0\cdot(r-\delta)QI + \delta\Delta - v_4 PC\cdot(1-\delta) \end{bmatrix}. \tag{15}$$

As $\gamma \to +\infty$, v_0 and v_2 go to $-\infty$ in such a way that $\frac{v_0}{v_2} \to 1$, while the remaining v_i remain bounded. Thus (15) converges in this case to the limiting form

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{PC}{QI} & 0 & \frac{r-\delta}{1-\delta} \\ \frac{-(1-\delta)PC}{QI} & 0 & r \end{bmatrix}, \tag{16}$$

which has real eigenvalues of *r* and 1. We conclude that in this limiting case of extreme risk aversion the Hall conclusion that consumption is approximately a random walk is correct, since the model has one explosive root, *r*, that will be suppressed in the solution, and one root of 1.

Another interesting case is $\mu = 1$, with no movement in the relative price of *C* and *I*. This makes $v_1 = v_3 = 0$. Then we can see from (15) and the definition of Δ that we will be back to the form (16) if $v_4 \to 0$. This occurs, as we can see by referring back to (12) (where v_4 is the coefficient on dk_t), when either $\sigma \to 1$ or $\alpha \to 1$, both of which deliver in the limit a linear production technology. But notice that linearity of *Y* in *K* is not enough. Generally if $\mu \neq 1$, the stable root of the system is not close to one even if $\sigma = 1$.

Note, though, that the discussion above has not checked whether a steady state exists in the limiting cases considered. In fact, they often do not. As a check on these analytical methods, therefore, we present below graphs of results of some numerical calculations. We will see that there is often no steady state when σ or α is near one.

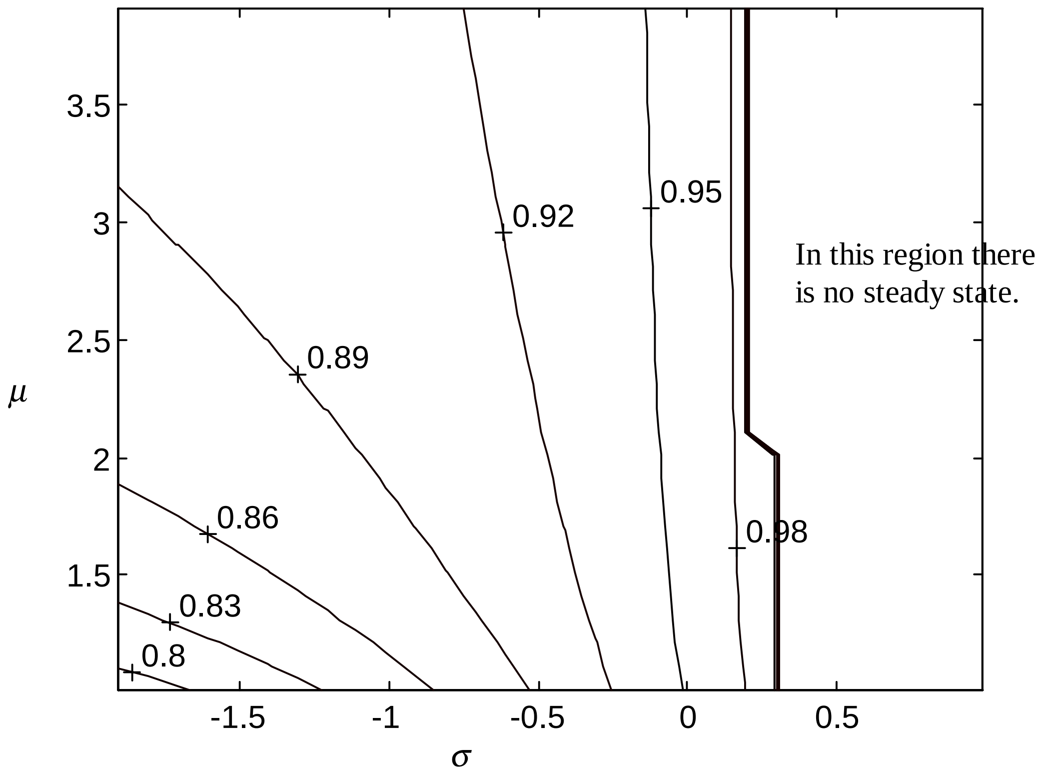
The first figure below shows contours for the single stable root of the linearized system as a function of μ and σ with $\gamma = 2$, $\delta = .93$, $\beta = .95$, $\theta = 1$, $\alpha = .5$. These parameters are chosen to be more or less realistic for an annual time unit. Note that there is a region in which the root is above .95 (even above 1 over part of the region). Most of the region is within the rectangle in which σ lies between -.1 and +.2, with the range of σ values in the region slightly widening as μ

increases. This result is quite sensitive to the choice of α. The second plot shows the effect of changing to $\alpha = .3$. In this plot roots drop away from one more rapidly as σ decreases. The band in which roots exceed .95 requires $\sigma > .2$. If we interpret actual income shares as reflecting competitive market determination of factor prices, the α value of .3 is more realistic. Values of α larger than the observed market share of capital are sometimes justified by a claim that some of observed labor income is actually a return to human capital. Of course if this is so, then a single-capital good model like this one will be a bad approximation, unless human and physical capital are nearly perfect substitutes.

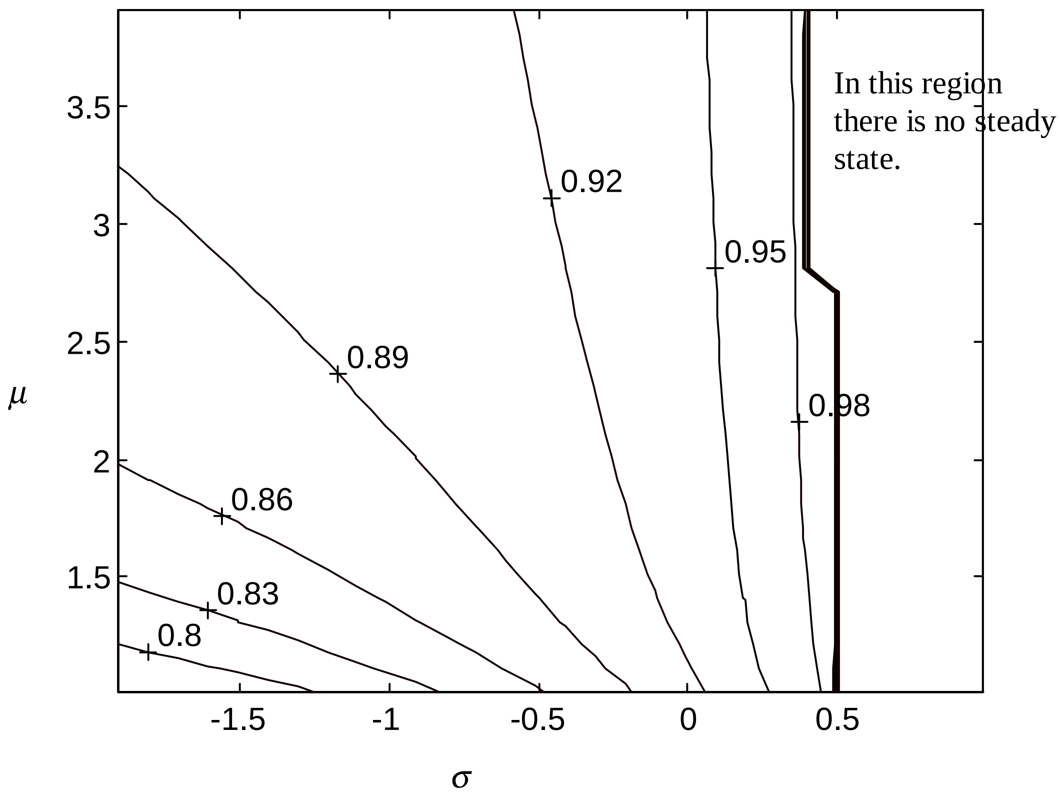
In the remaining figures other parameters are changed, in directions that make the martingale-like behavior of consumption an increasingly good approximation. By the last figure, we have made nearly the entire region where $\mu < 1.25$ or $\sigma > -.25$ display unit-root-like behavior. But to do this we have had to make γ higher than is usually thought to be realistic.

It seems fair to conclude from these plots that, with conventionally "reasonable" assumptions about risk aversion and the discount factor, the martingale-like behavior of consumption predicted by the permanent income hypothesis is a good approximation for aggregate data only when capital and labor are highly substitutable.

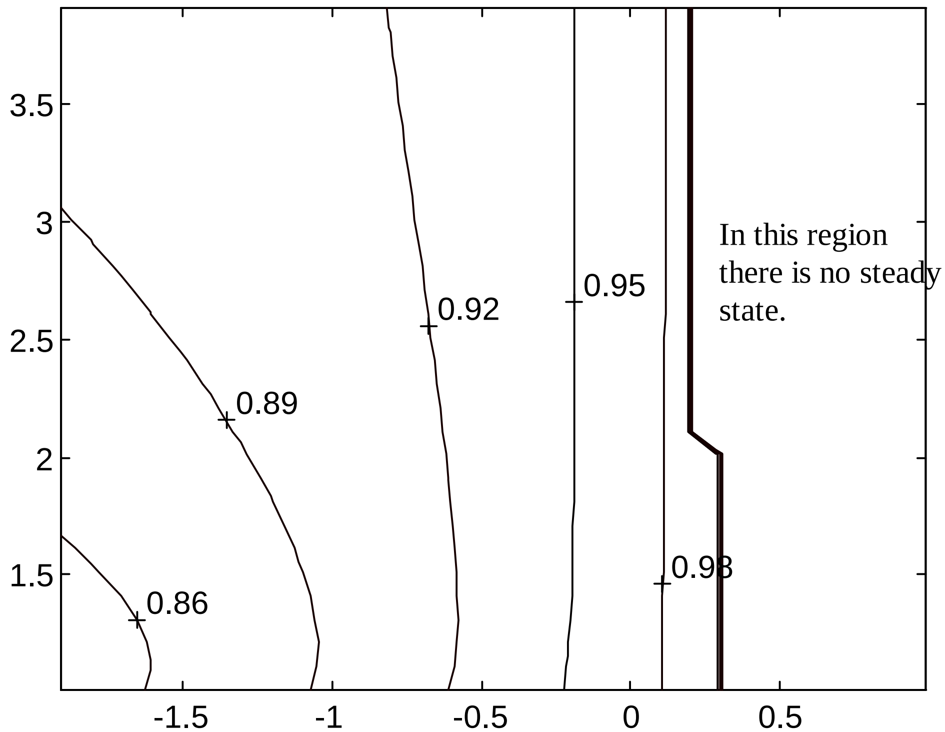
Stable Root as a Function of μ and σ, α=.5



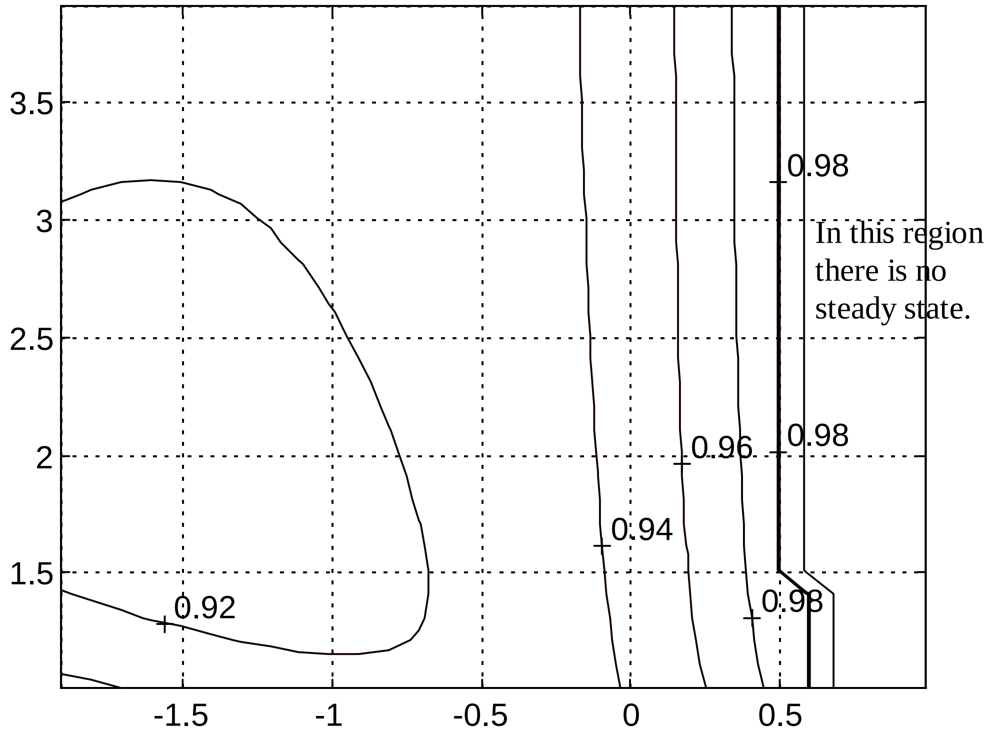
Stable Root as a Function of μ and σ, α=.3



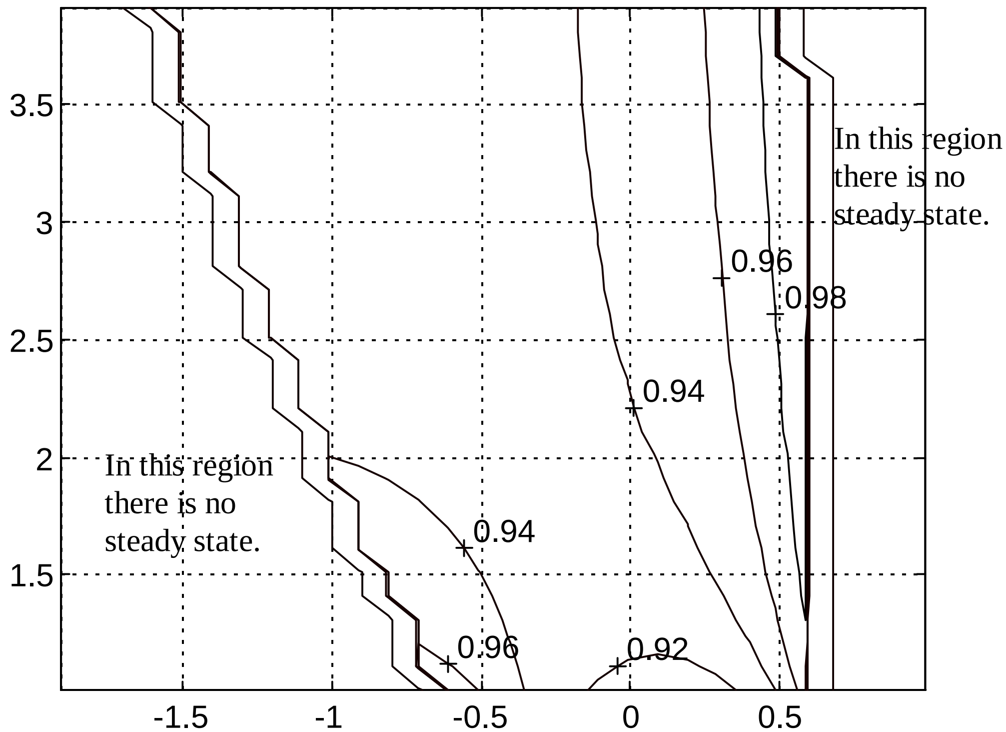
Stable Root as a Function of μ and σ, α=.5, γ=4



Stable Root as a Function of μ and σ, α=.3, γ=4, *A*=.5



Stable Root as a Function of μ and σ, α=.3, γ=2, A=.1



Stable Root as a Function of μ and σ, α=.3, γ=12, A=.5