Econ. 511b Spring 1998 G. Hall and C. Sims

Final Exam Answers, Problems 1 and 2

1.

a. The Euler equation FOC's are

∂C: $$\frac{1}{C_t} = \lambda_t \qquad [1]$$

∂B: $$\frac{\lambda_t}{P_t} = \beta R_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] \qquad [2]$$

∂L: $$2\gamma L_t = \alpha \lambda_t A_t L_t^{\alpha-1} \quad . \qquad [3]$$

The sufficient transversality condition is here that

$$\limsup_{t\to\infty} \beta^t E\left[\frac{B_t - B_t^*}{P_t C_t}\right] \le 0 \qquad [4]$$

for all B^* sequences that the agent sees as feasible.

b. By substituting [1] into [2] we arrive at

$$\frac{1}{P_t C_t} = \beta R_t E_t\left[\frac{1}{P_{t+1}C_{t+1}}\right] , \qquad [5]$$

which can easily be seen to be satisfied when $R_t = \beta^{-1}$ and $P_t C_t$ is constant. Using the government budget constraint ((2) on the exam) and the spending policy equation ((7) on the exam) together gives us the social resource constraint in the form

$$C_t \cdot \left(1 + \overline{G}\right) = A_t L_t^{\alpha} \ , \qquad [6]$$

while the FOC's [1] and [3] give us

$$C_t = \frac{\alpha A_t L_t^{\alpha-2}}{2\gamma} \ . \qquad [7]$$

We can satisfy both [6] and [7] if we set

$$L_t = \sqrt{\frac{\alpha \cdot \left(1+\overline{G}\right)}{2\gamma}} \ . \qquad [8]$$

So we can see that there is a solution to all the FOC's and the constraints in which L is constant and (from either [6] or [7]) C_t fluctuates proportionately to A_t. To check that the sufficient transversality condition is satisfied, note that because B is constrained to remain positive and $B/(PC)$ itself is constant in this equilibrium, the sufficient transversality condition is satisfied.

c. Following the hint, and using the notation $b = B/(PC)$, the GBC (equation (4) on the exam) becomes

$$b_t = \beta^{-1} b_{t-1} \frac{P_{t-1} C_{t-1}}{P_t C_t} + \overline{G} - \overline{\tau} \ . \qquad [9]$$

Taking E_{t-1} of both sides of this equation, using [5], we get the unstable difference equation in $E_{t-1} b_s$:

$$E_{t-1} b_t = \beta^{-1} b_t + \overline{G} - \overline{\tau} \ . \qquad [10]$$

A complete transversality argument showing that agents are not optimizing if $\beta^t b_t$ does not converge to zero was not expected on the exam, though it is fairly straightforward, as follows. The agent always has the option of consuming the entire real value of his holdings of government bonds in the current period, then from that point on working enough to make $C_t = 1$ and to pay taxes in every period. (This is possible because in this problem *L* is unbounded and production can be made arbitrary large by choosing *L* large enough.) This will deliver at any *t* some finite level of discounted expected utility over all dates $t+1$ and later, and because of the i.i.d. shocks this level is a constant. Furthermore the expected discounted utility over all dates $t+1$ and later from following the equilibrium consumption path given by [7] and [8] is also constant. But the utility gained in period *t* itself from consuming all the wealth is just $\log b_t$. Therefore if *b* becomes large enough, consuming all of *b* in the current period must appear to yield higher expected utility than the candidate equilibrium path.

So from [10] we conclude that there is only one possible value for *b*,

$$b_t = \overline{b} = \frac{\overline{\tau} - \overline{G}}{\beta^{-1} - 1} \ . \qquad [11]$$

But if *b* is constant at this value, we can then return to the original GBC to see that this constant value can be maintained only with a constant $P_t C_t$.

d. The answer to this part is included in the answer to (b) above.

e. Since $\overline{\tau} - \overline{G}$ does not change, we will still have the same level $\overline{b}$ for the ratio of real government debt to consumption. From our equation [8] for the level of *L*, we know that *L* will increase, and therefore from [7] that for every *t*, C_t will be lower, but less than proportionately to the increase in $1 + \overline{G}$. Thus the increased government purchases are supplied partly from increased labor, partly from decreased consumption. Since initial *B* is fixed and *b* is unchanged, *PC* is also unchanged, meaning that *P* increases in proportion to the *C* decrease.

2. Substituting out *r* using the Money Demand equation ((9) on the exam) gives us

$$\theta^{-1}\cdot\left(p_t+\varepsilon_t+\bar{y}-\bar{m}\right)=\bar{r}+p_{t+1}-p_t-\eta_{t+1} \qquad [12]$$

$$w_t=.5w_{t-1}+.25\left(p_t+p_{t+1}\right)-\xi_t+.25\eta_{t+1}\ , \qquad [13]$$

with $\eta_t=p_t-E_{t-1}p_t$. The first of these equations involves *p* and error terms alone. Since we are assuming that *p* (like the other variables) cannot explode rapidly, we can hope to analyze its stability based on [12] alone, which can be rewritten as

$$p_{t+1}=\left(1+\theta^{-1}\right)p_t+\theta^{-1}\cdot\left(\bar{y}-\bar{m}\right)-\bar{r}+\eta_{t+1}+\theta^{-1}\varepsilon_t\ . \qquad [14]$$

This appears explosive, because p_t on the right has a coefficient bigger than one, but we cannot conclude immediately that it is explosive, because the "error term" includes ε_t, which does not satisfy $E_t\varepsilon_t=0$. However, we can take E_{t-1} of both sides of the equation, to obtain a difference equation in $E_{t-1}p_s$:

$$E_{t-1}p_{t+1}=\left(1+\theta^{-1}\right)E_{t-1}p_t+\theta^{-1}\cdot\left(\bar{y}-\bar{m}\right)-\bar{r}\ . \qquad [15]$$

This is a non-stochastic, unstable difference equation. We know therefore that its only non-explosive solution is

$$\bar{p}=E_{t-1}p_t=\bar{m}-\bar{y}+\theta\bar{r}\ . \qquad [16]$$

Now we can use this solution in the original [12] to obtain

$$p_t=\frac{\theta}{1+\theta}\left(\bar{r}+\bar{p}\right)+\frac{\bar{m}-\bar{y}-\varepsilon_t}{1+\theta}\ . \qquad [17]$$

This makes p_t i.i.d., fluctuating inversely with money demand shocks.

To find the implied behavior of *w* (but note that this was not asked for on the exam) we again use [16], substituting it this time into (10) from the exam (the wage adjustment equation) to obtain

$$w_t=.5w_{t-1}+.25\left(p_t+\bar{p}\right)-\xi_t\ . \qquad [18]$$

This is a stable difference equation, determining w_t from current and past values of *p* and ξ.

The brute force approach to this problem would apply the matrix methods discussed in class and the notes. The appearance of three time subscripts in (10) on the exam means that to apply the matrix methods we must introduce a dummy variable $z_t=E_tp_{t+1}$ and introduce its definition as a third equation. If we order the variables as *p*, *z*, *w*, and stack the equations with the definition of *z* on top, with [12] and [13] below, the system is in the canonical form

$$\Gamma_0y_t=\Gamma_1y_{t-1}+\Psi\begin{bmatrix}\varepsilon_t\\ \xi_t\end{bmatrix}+\Pi\eta_t+C \qquad [19]$$

if we set

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1+\theta^{-1} & -1 & 0 \\ -.25 & -.25 & 1 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .5 \end{bmatrix} \tag{20}$$

$$\Psi = \begin{bmatrix} 0 & 0 \\ -\theta^{-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \Pi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ -\theta^{-1} \cdot (\bar{y} - \overline{m}) + \bar{r} \\ 0 \end{bmatrix} . \tag{21}$$

Then, because Γ_0 is triangular and Γ_1 is so sparse, it is relatively easy to compute

$$\Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1+\theta^{-1} & 0 \\ 0 & .5+.25\theta^{-1} & .5 \end{bmatrix} , \tag{22}$$

which is easily seen to have two real characteristic roots, a stable one of .5 and an unstable one of $1+\theta^{-1}$. It is also easy to verify that the left eigenvector associated with the unstable root is just [0 1 0]. Multiplying the whole system through by

$$[0 \quad 1 \quad 0]\,\Gamma_0^{-1} = \begin{bmatrix} 1+\theta^{-1} & -1 & 0 \end{bmatrix} \tag{23}$$

(note that the second row of Γ_0 and of Γ_0^{-1} happen to be the same) produces

$$z_t = \left(1+\theta^{-1}\right) z_{t-1} + \theta^{-1}\varepsilon_t + \left(1+\theta^{-1}\right)\eta_t + \theta^{-1} \cdot (\bar{y} - \overline{m}) - \bar{r} \ . \tag{24}$$

Imposing stability on this equation requires that z_t be constant, and thereby leads us to the same conclusions that we reached above by the first approach.