Answer to MidTerm Question 2

The statement of this problem included an error. The decision rule for the problem in standard form, without the $C_t \leq W_t$ constraint and with a two-sided limit on the growth rate of W, was given as $C_t = \left(\beta^{-1} - 1\right)W_t$, when in fact the correct form of the decision rule is $C_t = (1-\beta)W_t + \beta\overline{Y}$, where $\overline{Y}$ is the mean of Y_t. There is a way of doing the problem, described below (the way I did it myself) for which this error is inconsequential. There is another approach, taken by most people who at least started off in the right direction, in which the error in the problem statement caused serious difficulties.

Any correct analysis has to start with first order conditions for the problem with inequalities. They are

∂C_t:
$$1 - C_t = \beta \cdot (1+r)E_t\lambda_{t+1} + \mu_t \qquad [1]$$

∂W_t:
$$\lambda_t = \beta \cdot (1+r)E_t\lambda_{t+1} + \mu_t \qquad [2]$$

with λ_t and μ_t the Lagrange multipliers on (7) and (10) from the problem statement, respectively. As usual, the Lagrange multipliers are positive when the constraints bind and zero otherwise. (This depends on their being written with $\leq$ rather than $\geq$, of course.) These two FOC's can be solved to eliminate λ, resulting in (using the condition you were given that $\beta \cdot (1+r) = 1$)

$$1 - C_t = \lambda_t \qquad [3]$$

$$C_t = E_t C_{t+1} - \mu_t \ . \qquad [4]$$

[One incorrect approach to an answer was to observe that since μ is always non-negative, [4] implies that C_t is lower than it would be in an equilibrium without the constraint. The fallacy is that C_{t+1} will also be different from what it would be in the other equilibrium. We are interested in whether C is lower or higher at a given level of the state W, not at a given level of $E_t C_{t+1}$.]

One correct approach goes on from here to use [4] in the budget constraint (7) and solves forward. This results in

$$W_t = E_t W_t$$
$$\geq E_t\left[\sum_{s=0}^{T} C_{t+s} \cdot (1+r)^{-s} - \sum_{s=1}^{T+1}(1+r)^{-s}Y_{t+s} + (1+r)^{-T-1}W_{t+s}\right] \qquad [5]$$
$$= \frac{C_t}{1-\beta} + E_t\left[\sum_{s=0}^{\infty}\beta^s\mu_{t+s}\right] - \frac{\beta\overline{Y}}{1-\beta} + \lim_{T\to\infty}\beta^{T+1}E_t W_{t+T+1}$$

The last term does *not* necessarily go to zero as $T \to \infty$ in this model. The second and last terms on the right of [5] are both non-negative and not present in the standard model. Hence C is always no higher, at a given level of W. As C approaches satiation, it may be optimal to let W grow without bound at the rate $1+r$. However at low levels of W, when C is far below 1, the time when this growth starts to happen will be far off and the last term may be small. But it is exactly when W is low that it becomes likely that the second term on the right of the last row of [5] is substantial, because then μ's in the near, not-much-discounted future are positive with high probability. When W is very high, this second term may be small or zero, because the probability of (10) binding in the near future is very small or zero. However then one of two things will be true. One possibility is that W will be so high that even satiation consumption does not absorb much of the earnings from wealth, so that W is growing at approximately the rate $1+r$ and the last term in [5] is substantial. The other possibility is that C has actually hit the satiation level. In this case the budget constraint is an inequality, all future μ's are known to be zero, and, if excess W is discarded (W is actually indeterminate because of the inequality), the last term in [5] is zero. But the fact that the budget constraint is an inequality now tells us directly that C is less than would be predicted from the previous model.

The other approach to a correct answer simply notes that we can see directly that it is never optimal to let C exceed one. Since the standard model makes C linear in W, it must at large enough W imply C greater than one, even much greater than one, so at such levels of wealth the model with inequalities clearly implies smaller C. At levels of W close to zero, (10) implies that C must get arbitrarily close to zero. With the correct form of the decision rule, one can conclude that, since the standard model implies $C_t = (1-\beta)W_t + \beta\overline{Y}$, it will for small enough W imply $C_t > W_t$, because of the assumption that $\overline{Y} > 0$. Of course if you used the incorrect version of the decision rule given in the problem statement and took this approach, you would have run into trouble at this point because the incorrect rule does imply that C approaches zero as W approaches zero.

Answers that got the FOC's correct for the inequality constrained model received about half credit for just that. Answers that started from there in a reasonable direction and ran afoul of the mistake in the question got an additional 3 or 4 points. One answer, that used the satiation argument for high W and ran afoul of the bad decision rule only for low W, earned 17/20.