Decentralizing with Incomplete Asset Markets: No Firms
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I. Introduction

In a competitive model with complete markets, agents will tend to insure themselves against wealth fluctuations. In general, when markets are incomplete agents are unable to prevent stochastic disturbances from affecting their wealth more than would be true with complete markets, and this has first-order effects on the solution. These notes discuss a symmetrically decentralized model, with two types of consumers (and, to keep things simple, no firms). Here a single-traded-asset equilibrium generally does not reproduce complete-markets equilibrium, and we can see the mechanism by which incomplete markets force deviation from a complete markets solution.

II. The Model

The model has two types of agents, indexed by i=1,2, each endowed with an i.i.d. stochastic stream of income Y_{it}. The income streams of the two types have the same distribution, but they are independent of each other. Both agents have the same discount factor and within-period utility function. There is no form of physical investment available; each period, all of the endowment has to be consumed. The objective function of agent i is

$$\max E\left[\sum_{t=0}^{\infty}\beta^t U\left(C_{it}\right)\right] \tag{1}$$

The social resource constraint is

$$C_{1t}+C_{2t}=Y_{1t}+Y_{2t}\text{, all } t. \tag{2}$$

III. Planner's, or Complete Market Solution

A planner that weights the utilities of the two agents equally has a very simple problem to solve. There is no opportunity to move resources across time, so the planner simply maximizes the sum of utilities at each date. Since the two utility functions are the same, this results in

$$C_{1t}=C_{2t}=\frac{Y_{1t}+Y_{2t}}{2}\text{, all } t. \tag{3}$$

It is possible to duplicate this solution with a decentralized model of asset trading, if the right assets are available for trade. With continuously distributed Y's (i.e., with something like a normal or gamma distribution for the Y's) we would expect in general (e.g. if the two agents had different nonlinear utility functions) to have to trade infinitely many securities at each date to achieve the complete markets solution. However, this model has such a simple structure that it turns out we can achieve the complete markets solution with two securities: One that pays a dividend proportional to Y_{1t}, and another that pays a dividend proportional to Y_{2t}. Then agent 1 will sell agent 2 some of the security that pays Y_1 as a dividend in return for some of the security that pays Y_2 as a dividend. They will trade this way to achieve a situation where each has total income, summing endowment income and net income from securities bought and sold, that is proportional to $Y_{1t} + Y_{2t}$, and they will consume exactly their incomes, so that after the initial asset trades, no further asset trading occurs.

[It is left as an exercise for you to verify that the equilibrium described above is in fact a competitive equilibrium. It is quite easy to do this if one assumes that at time t=0 each agent starts out having in effect sold half of his endowment stream to the other agent for half of the other agent's endowment stream. A little more interesting is what happens if at t=0 neither agent begins with any assets. Since at t=0 each agent knows Y_{10} and Y_{20}, they do not begin a symmetric situation – one will in general have had a luckier draw than the other from the endowment process. They will still trade assets at time 0 in such a way that neither ever desires to trade again after time 0 and in such a way that each has C_{it} proportional to $Y_{1t} + Y_{2t}$. However the proportions will not in general be equal. Determining how these proportions depend on the values of the Y_{i0}'s is a good exercise.]

IV. Symmetric Bonds-Only Equlibrium

But now suppose there is only a single asset. We will assume at first that it is a standard one-period loan or bond contract, paying a rate of interest fixed at the time the asset is issued. There is no government to issue this debt, so any bonds held by one agent must have been issued by the other agent. The constraints of the two agent types are given by

$$C_{1t} + B_t = Y_{1t} + r_{t-1}B_{t-1}\,,\; B_t \geq -H \tag{4}$$

$$C_{2t} - B_t = Y_{2t} - r_{t-1}B_{t-1}\,,\; -B_t \geq -H \;\; . \tag{5}$$

The opposite signs on the B terms in these constraints reflect the fact that B represents bonds purchased by agent 1 from agent 2. When B is negative, agent 1 is borrowing from agent 2 instead of lending. Clearly adding the first equations of (4) and (5) gives us the original social resource constraint (2).

First note that it is impossible to implement complete markets solutions with trading only in B. In the complete markets solution, $C_{1t} = C_{2t}$ for all t, so that when we subtract (5) from (4) we obtain

$$B_t = r_{t-1}B_{t-1} + \frac{Y_{1t} - Y_{2t}}{2}\,. \tag{6}$$

We also have first order conditions from the individual maximization problems, of the form

$$U'(C_{it}) = \beta r_t E_t\big[U'(C_{i,t+1})\big]\;. \tag{7}$$

Using the fact that the Y's are i.i.d. and that in the $C_{1t} = C_{2t}$ solution each C is just half of the total endowment, (7) implies that

$$E_{t-1} r_t = \beta^{-1} . \tag{8}$$

Using (6) and (7) or (8) together we can show that B must in a sense explode at the rate β^{-1}. There are (at least) two ways to do this. One approach, which works on a wide class of models, is to modify (6) so that we can apply (7) to it. Multiplying it on both sides by $U'(C_{it})$ and taking E_{t-1} of both sides produces

$$E_{t-1}\left[U'(\bar{Y}_t)B_t\right] = r_{t-1}E_{t-1}\left[\frac{U'(\bar{Y}_t)}{U'(\bar{Y}_{t-1})}\right]U'(\bar{Y}_{t-1})B_{t-1} + E_{t-1}\left[\frac{Y_{1t}-Y_{2t}}{U'(\bar{Y}_t)}\right] = \beta^{-1}U'(\bar{Y}_{t-1})B_{t-1} , \tag{9}$$

where we have used the fact that we are considering the allocation in which $C_{1t} = C_{2t} = (Y_{1t}+Y_{2t})/2 = \bar{Y}_t$, the fact that with $\bar{Y}_t$ defined this way $E\left[Y_{1t}-Y_{2t} \middle| \bar{Y}_t\right] = 0$ (because the Y_{it}'s are independent and identically distributed), and (7). The far left and far right components of (9) give us an explosive difference equation in $E_{t-1}\left[B_{t+s}U'(\bar{Y}_{t+s})\right]$.

The other possible approach is to observe that (6) implies that B_t is a linear function of $Y_{1t}-Y_{2t}$, that (7) implies that the randomness in r_t is a function of $\bar{Y}_t$ alone, and that since $E\left[Y_{1t}-Y_{2t} \middle| \bar{Y}_t\right] = 0$,

$$E_{t-1}\left[B_t \middle| r_t\right] = r_{t-1}B_{t-1} . \tag{10}$$

Using (8) and (10) together recursively allows us to write

$$E_t B_{t+s} = \beta^{-s+1} r_t B_t \ . \tag{11}$$

In other words, B explodes at the rate β^{-t}, even if there are no random disturbances to the model, unless $B \equiv 0$. If B explodes, then it hits the $|B| < H$ constraint in finite time, after which it will be impossible to maintain $C_{1t} = C_{2t}$. And if B is identically zero, then $C_{it} \equiv Y_{it}$, which also contradicts the complete-markets solution assumption.

But there is nonetheless a competitive solution in this framework. Deriving it exactly, taking account of the $|B| < H$ constraint, is hard. But we can fairly easily find a linearized solution about a deterministic steady state with $C_1 = C_2 = Y_1 = Y_2$ and $B = 0$. The equation system we will consider is formed by (2), (4), and (7) for i=1,2. The four endogenous variables are C_1, C_2, B, and r. The exogenous random disturbances are Y_1 and Y_2, and endogenous expectational errors occur in the two versions of (7). Recall that our notes on linear rational expectations models use the standard notation

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + C + \Psi\varepsilon_t + \Pi\eta_t , \tag{12}$$

where y is the vector of variables in the system, ε is the vector of exogenous random disturbances, and η is the vector of endogenous forecast-error random terms. Linearizing our four-equation system and using this standard notation, ordering the variables as C_1, C_2, r, B, leads to

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -U'' & 0 & 0 & 0 \\ 0 & -U'' & 0 & 0 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1} \\ -U'' & 0 & \beta U' & 0 \\ 0 & -U'' & \beta U' & 0 \end{bmatrix}, \tag{13}$$

$$\Psi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{14}$$

Here we interpret the variables all as deviations from steady-states, which leads to C=0 in the linearized system.

Four-dimensional systems can easily be too messy to make hand algebra computations worthwhile, but here the system has enough simple structure to make hand calculations useful. With a singular Γ_0 as here, an approach that sometimes works is to find the linear combination of equations (i.e., the left eigenvector of Γ_0) that corresponds to the singularity, derive from it an exact contemporaneous relation among the variables, and use that to reduce the number of variables in the system. With this approach, though, one has to decide which equation to drop as the dimension of the system is reduced. Sometimes it doesn't matter which equation is dropped, but in some cases – including this model – it does. A safer approach that is really no more work, is based on finding the generalized eigenvalues and eigenvectors of the system, and we will now apply this approach.

The overall strategy is to find a non-singular matrix Q and a pair of diagonal matrices Λ_0 and Λ_1 such that

$$\Lambda_0 Q \Gamma_0 = \Lambda_1 Q \Gamma_1 \ . \tag{15}$$

The rows of the matrix Q are the left generalized eigenvectors of the pair Γ_0, Γ_1, and the ratios of the diagonal elements of Λ_0 to the corresponding elements of Λ_1 (which will be infinite in some cases when Γ_0 is singular) are the generalized eigenvalues of the pair. This approach is simpler than the *QZ* decomposition method applied in the `gensys` program, but it breaks down if there are repeated eigenvalues, or even nearly equal eigenvalues. In such cases hand algebra is unlikely to be helpful, so an approach that assumes away such complications is appropriate when we are attempting to solve a system by hand. If we can find the Q and Λ's in (15), then we can multiply the system (12) by $\Lambda_0 Q$ to obtain

$$\Lambda_0 Q \Gamma_0 y_t = \Lambda_1 Q \Gamma_1 y_t = \Lambda_0 Q \Gamma_1 y_{t-1} + \Lambda_0 Q \Psi \varepsilon_t + \Lambda_0 Q \Pi \eta_t \ . \tag{16}$$

(We have omitted the constant term here, as it is not present in the problem at hand.) If we introduce the transformed variable $w_t = Q\Gamma_1 y_t$, the system becomes

$$\Lambda_1 w_t = \Lambda_0 w_{t-1} + \xi_t\,, \tag{17}$$

where ξ_t is a disturbance term made up of the last two additive terms in (16). This is a set of univariate different equations, because the Λ's are diagonal, and as usual, if the ξ's have zero conditional expectations given past information, to guarantee stability we need to set to zero any elements of the *w* vector that correspond to explosive generalized eigenvalues. Note that when Γ_1 is singular, one or more elements of the *w* vector will be zero by construction, for which the corresponding diagonal element of Λ_0 is also zero. The equation system (17) will therefore contain some trivial "0=0" equations.

To find by hand the generalized eigenvalues and left generalized eigenvectors, one begins by solving the equation

$$\left|\Gamma_0 - \lambda\Gamma_1\right| = 0\,. \tag{18}$$

The roots of this equation will give the inverses of all the generalized eigenvalues except those corresponding to zero eigenvalues of Γ_1. In particular, all the unstable eigenvalues will emerge as roots of (18) less than one in absolute value. In our problem, this equation becomes

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1-\lambda\beta^{-1} \\ -U''\cdot(1-\lambda) & 0 & -\lambda\beta U' & 0 \\ 0 & -U''\cdot(1-\lambda) & -\lambda\beta U' & 0 \end{vmatrix} = 0\,. \tag{19}$$

Because the last column has only one element, this determinant is easily calculated as

$$\left(1-\beta^{-1}\lambda\right)(1-\lambda)\left(-2\lambda\beta U'U''\right)\,, \tag{20}$$

which is easily seen to have roots of 0, 1 and β. (Since there are only three, in this four-dimensional problem, we know Γ_1 has one zero eigenvalue.) The roots of 0 and β correspond to unstable roots of the system and must be suppressed for a stable solution.

To find the corresponding left eigenvectors, we solve the equation

$$c'\Gamma_0 = \lambda c'\Gamma_1 \tag{21}$$

for c for $\lambda = 0, \beta$. The two left eigenvectors turn out to be $\begin{bmatrix} 1 & 0 & 1/U'' & 1/U'' \end{bmatrix}$ and $\begin{bmatrix} -U''\cdot(1-\beta) & 2U''\cdot(1-\beta) & 1 & -1 \end{bmatrix}$, respectively. Note that these are not themselves the weights on *y* in the stability conditions. Those are found by finding the corresponding elements of *w*, i.e. by premultiplying Γ_1 by the eigenvectors. This produces two equations:

$$dC_{1t} + dC_{2t} = \frac{2\beta U'}{U''}\,dr_t \tag{22}$$

$$dC_{1t} - dC_{2t} = \frac{2(1-\beta)}{\beta}\,dB_t\,. \tag{23}$$

(Here the "d"'s indicate deviations from steady state.) Equation (22) asserts an exact contemporaneous relation ship between total consumption (which is also total endowment income) and the interest rate, with high consumption associated with low (because $U'' < 0$) interest rates. Equation (23) asserts that increased relative wealth of agent 1 leads to higher consumption for agent 1, with the amount of the increase corresponding to difference in net interest income between the two agents. Equation (23) characterizes the difference between this incomplete-markets equilibrium and the complete-markets allocation. A consumer who has had relatively bad luck (relatively low Y) will borrow to avoid reducing consumption by the full amount of the adverse shock to current income, but because the borrowing reduces the consumers wealth, the adverse shock is not fully offset as it would be in a complete markets solution.

The element of w corresponding to the unit root is simply $dC_{1t} - dC_{2t}$. That is, in this solution the levels of consumption of the two agents drift apart from each other. In the light of (23), this corresponds to a drift in the level of debt B. This does not invalidate our linear approximation, because with shocks small enough the drift will take arbitrarily long to get far from the steady state. However it does show that the linear approximation cannot be good forever. Eventually debt will reach levels where the borrowing constraint is binding, or likely soon to be binding, on one agent or the other, at which point taking full account of the non-linearities in the model becomes essential.

To be sure that the solution we are discussing exists and is unique, we need to check the implications of the stability conditions for relations among the error terms. The two equations relating error terms that arise from the stability conditions, formed by setting to zero the rows of $Q\Psi\varepsilon_t + Q\Pi\eta_t$ corresponding to unstable roots, are

$$-U'' \cdot (1-\beta)(dY_1 - dY_2) = \eta_1 - \eta_2 \tag{24}$$

$$dY_1 + dY_2 = -\frac{\eta_1 + \eta_2}{U''} \ . \tag{25}$$

It is easy to see that, so long as $U'' \neq 0$, these equations define a one-one relation between the Y's and the η's, so the solution exists and is unique.

This model can be interpreted as a schematic model of international borrowing and lending as well as of consumption-smoothing among individuals in incomplete asset markets. In both cases we see that there is first-order deviation between the complete markets solution and one with borrowing and lending only, and also that for a complete solution eventually taking account of the complications implied by borrowing constraints or bankruptcy will be necessary, even if one begins from a situation with no debt.

V. Differences in Risk Aversion

Now we consider some variations on the model. If the two types of agent differ in risk aversion, then they will use asset markets to allow the more risk averse agent to have less variable consumption, though slower expected growth in wealth. Rather than repeating the analytic derivation above, we present below a computer calculation of he results. Recall that the variables are ordered as C_1, C_2, R, B. We take B to be zero at the point around which we linearize and give Y_1 and Y_2 (and hence C_1 and C_2) means of 1.

```
gamma1 =
     1
gamma2 =
     2
» g0=[1 1 0 0;1 0 0 1;-gamma1 0 0 0;0 -gamma2 0 0]
g0 =
     1     1     0     0
     1     0     0     1
    -1     0     0     0
     0    -2     0     0
» g1=[0 0 0 0;0 0 0 1/.95 ;-gamma1 0 .95 0;0 -gamma2 .95 0]
g1 =
            0            0            0            0
            0            0            0       1.0526
           -1            0         0.95            0
            0           -2         0.95            0
» C=[0;0;0;0];psi=[1 1;1 0;0 0;0 0]
psi =
     1     1
     1     0
     0     0
     0     0
» pi=[0 0;0 0;eye(2)]
pi =
     0     0
     0     0
     1     0
     0     1
» [G1,C,impact,fmat,fwt,ywt,gev,eu]=gensys(g0,g1,C,psi,pi,1.05);
» eu
eu =
     1
     1
» impact
impact =
      0.68333      0.63333
      0.31667      0.36667
      0.70175      0.70175
      0.31667     -0.63333
» G1*impact
ans =
     0.016667    -0.033333
    -0.016667     0.033333
  6.2309e-017 -1.2462e-016
      0.31667     -0.63333
» G1*ans
ans =
     0.016667    -0.033333
    -0.016667     0.033333
  6.2309e-017 -1.2462e-016
      0.31667     -0.63333
```

These calculations show that C_1, the consumption of the first, less risk-averse agent responds in equilibrium about twice as much as does C_2, regardless of whether the shock is to the agent's own income or to the other agent's income. However, because of market incompleteness, each agent responds slightly more to a shock in his own income than the other agent does. There is the same unit root present as in the symmetric equilibrium, so once again it is clear that eventually bankruptcy or limits on borrowing will become relevant.

Both this solution and that of the previous section show that even with only bonds available, competitive equilibrium comes close to spreading income shocks evenly across agents. This result is a consequence of our having considered so far only the case of i.i.d. Y's. It is feasible for agents to cushion temporary income shocks by borrowing and lending to each other. But if the shocks are highly persistent, they have a stronger effect on wealth, and borrowing and lending may not provide much of a cushion. Here is a version of the model in which we have added

$$Y_{it} = .1 + .9Y_{it-1} + \varepsilon_{it},\ i = 1,2\ . \tag{26}$$

```
» g0=[1 1 0 0 -1 -1;1 0 0 1 -1 0;-gamma1 0 0 0 0 0;0 -gamma2 0 0 0 0;
   0 0 0 0 1 0;0 0 0 0 0 1]
g0 =
     1     1     0     0    -1    -1
     1     0     0     1    -1     0
    -1     0     0     0     0     0
     0    -2     0     0     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
» g1=[0 0 0 0 0 0;0 0 0 1/.95 0 0;-gamma1 0 .95 0 0 0;
   0 -gamma2 .95 0 0 0;zeros(2,4) .9*eye(2)]
g1 =
            0            0            0            0            0            0
            0            0            0       1.0526            0            0
           -1            0         0.95            0            0            0
            0           -2         0.95            0            0            0
            0            0            0            0          0.9            0
            0            0            0            0            0          0.9
» C=[0;0;0;0;.1;.1];psi=[0 0;0 0;0 0;0 0;eye(2)]
psi =
     0     0
     0     0
     0     0
     0     0
     1     0
     0     1
» pi=[0 0;0 0;eye(2);zeros(2)]
pi =
     0     0
     0     0
     1     0
     0     1
     0     0
     0     0
» [G1,C,impact,fmat,fwt,ywt,gev,eu]=gensys(g0,g1,C,psi,pi,1.05);
» eu
eu =
     1
     1
» impact
impact =
      0.78161      0.43678
      0.21839      0.56322
     0.070175     0.070175
      0.21839     -0.43678
```

```
             1  1.0408e-016
  1.0384e-017            1
» G1*impact
ans =
      0.71494      0.37011
      0.18506      0.52989
     0.063158     0.063158
      0.41494     -0.82989
          0.9  2.0275e-016
  5.9144e-017          0.9
» G1*ans
ans =
      0.65494      0.31011
      0.15506      0.49989
     0.056842     0.056842
      0.59184      -1.1837
         0.81  2.8015e-016
  9.8239e-017         0.81
» gev(:,1).\gev(:,2)
ans =
           0
           1 -3.5947e-016i
         0.9 -2.8382e-017i
         0.9
 5.2392e+015 - 1.379e+014i
      1.0526 +3.3746e-016i
```

The shocks are now less smoothed across agents, and they have persistent impacts, not only on consumption, but more strongly so on relative wealth.

VI. Assets with Random Returns

The failure of the market here to deliver a Pareto optimum is not simply a matter of not having an asset with a stochastic return. An asset that delivers a return of $Y_1 + Y_2$, for example, leaves us with no improvement in the equilibrium, as it still provides no way to exchange insurance against idiosyncratic income shocks. An asset that delivers a return of $Y_1 - Y_2$ does much better, at least when risk aversions are similar. To discuss such assets, we need to modify the notation. We replace (4) and (5) with

$$C_{1t} + Q_t S_t = Y_{1t} + (Q_t + \delta_t) S_{t-1}\,,\ Q_t S_t \geq -H \tag{27}$$

$$C_{2t} - Q_t S_t = Y_{2t} - (Q_t + \delta_t) S_{t-1}, -Q_t S_t \geq -H \tag{28}$$

The FOC's (7) become

$$Q_t U_i'(C_{it}) = \beta E_t\left[U_i'(C_{i,t+1})(Q_{t+1} + \delta_{t+1})\right],\ i = 1,2\ . \tag{29}$$

Let us consider first the security which pays $\delta_t = Y_{1t} + Y_{2t}$. The linearized system takes the form

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & Q \\ -U''\beta^{-1}Q & 0 & -\beta U' & 0 \\ 0 & -U''\beta^{-1}Q & -\beta U' & 0 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1}Q \\ -U''Q & 0 & -U' & 0 \\ 0 & -U''Q & -U' & 0 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ \beta U' & \beta U' \\ \beta U' & \beta U' \end{bmatrix}, \ \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$