Econ. 511b Spring 1998 C. Sims

# Notes: Fiscal Theory of the Price Level

## I. A model with prices but without money

Agents solve this problem:

$$\max_{\{C_t, B_t\}} E\left[\sum_{t=0}^{\infty} \beta^t \log C_t\right] \tag{1}$$

subject to

$$C_t + \frac{B_t}{P_t} = \frac{R_{t-1}B_{t-1}}{P_t} + Y_t - \tau_t \tag{2}$$

$$B_t \geq 0 \,. \tag{3}$$

$Y_t$ is a positive random variable, i.i.d. across time. To simplify some arguments below, we assume that it has a lower bound $\overline{Y} > 0$.

The government budget constraint is

$$B_t + P_t\tau_t = R_{t-1}B_{t-1} \ . \tag{4}$$

The government must fix two of the four variables in (4), or else two functions of them, in order for the model to be complete. The other two are then determined by (4) and the FOC's of the private agents.

The private agents' FOC's are

$\partial C$:

$$\frac{1}{C_t} = \lambda_t \ . \tag{5}$$

$\partial B$:

$$\frac{\lambda_t}{P_t} = \beta R_t E_t \frac{\lambda_{t+1}}{P_{t+1}} \tag{6}$$

which reduce to

$$\frac{1}{C_t P_t} = \beta R_t E_t \frac{1}{P_{t+1}C_{t+1}} \ . \tag{7}$$

Suppose the government fixes $R$ and $\tau$ at constant values. For simplicity, we assume that $\tau < \overline{Y}$, so that the individual can always pay taxes even when wealth in form of government debt is zero. Then the government budget constraint (4), divided through by $P_tC_t$, is given by

$$\frac{B_t}{P_tC_t} = R\frac{P_{t-1}C_{t-1}}{P_tC_t}\cdot\frac{B_{t-1}}{P_{t-1}C_{t-1}} - \frac{\tau}{C_t} \ . \tag{8}$$

Taking $E_{t-1}$ of (8), using (7) and the facts that $C_t = Y_t$ in equilibrium and $Y_t$ is i.i.d., we get

$$E_{t-1}\left[\frac{B_t}{P_t Y_t}\right] = \text{b}^{-1}\frac{B_{t-1}}{P_{t-1}Y_{t-1}} - \text{t}\, E\left[Y_t^{-1}\right] \ . \tag{9}$$

This is an unstable difference equation. Any solution to (9) contains a component in which $E_{t-1}\left[B_{t-1+s}\big/\left(P_{t-1+s}C_{t-1+s}\right)\right]$ grows at the rate $\text{b}^{-t}$, except the constant solution that sets

$$\frac{B_t}{P_t Y_t} = \frac{\text{b}\cdot\text{t}}{1-\text{b}} E\left[Y_t^{-1}\right] \ . \tag{10}$$

We will come back below to the question of whether we know $\dfrac{B}{PY}$ must remain bounded. With $\dfrac{B}{PY}$ determined as constant by (10), we can use (8) to determine $P_t$ from knowledge of $Y_t$. The result is

$$P_t = \frac{RP_{t-1}Y_{t-1}\text{x}}{\text{x}Y_t + \text{t}} \ , \tag{11}$$

where x is the constant value of $B/PY$. Note that *P* moves inversely with current *Y*, though less than fully proportionately. [There is a steady state in which the behavior of *P* is somewhat easier to characterize when instead of fixing t as a constant, the government sets $\text{t}_t = \text{f}\, Y_t$. You should be sure you can find and characterize this steady state.]

Back to the question of how we know that $B/PY$ cannot grow as fast as $\text{b}^{-t}$. The full transversality condition as laid out in the updated "Random Lagrange Multipliers" notes is, as applied to the private agent in this model,

$$\liminf_{T\to\infty} \text{b}^T E\left[\frac{\text{l}_T}{P_T}\cdot\left(B_T^* - \overline{B}_T\right)\right] \geq 0 \tag{12}$$

for any privately feasible $B_T^*$ path, where $\overline{B}_T$ is our candidate equilibrium path. On our candidate path on which $\text{l}_T\, \overline{B}_T / P_T = \overline{B}_T / \left(P_T Y_T\right)$ is constant, (12) becomes simply

$$\liminf_{T\to\infty} \text{b}^T E\left[\frac{B_T^*}{P_T Y_T}\right] \geq 0 \ . \tag{13}$$

Since the term in brackets is always positive, this condition is obviously met. Notice that in checking this condition we kept *P* and *C* at their equilibrium time paths, particularly *C*=*Y*, while considering individual deviation of *B* from its original time path. This is because the derivatives that enter the transversality condition are evaluated at the equilibrium solution. There is in principle a second component of the transversality condition corresponding to *C*, the other choice variable of the agent. However here, as usual with choice variables that enter only contemporaneously (and not also as lagged variables), the transversality condition corresponding to *C* is empty. It simply states

$$\limsup_{T\to\infty} \beta^t E\left[\left(\frac{1}{\overline{C}_t} - \lambda_t\right)\cdot\left(C_t^* - \overline{C}_t\right)\right] \leq 0\,, \tag{14}$$

and what appears in brackets is already guaranteed, by the Euler equation FOC's, to be identically zero.

Now consider a different candidate optimum, in which initial $\overline{B}_0 \big/ \left(\overline{P}_0 Y_0\right) \neq \xi$. It cannot be below $\xi$, because that would imply via (9) a non-zero probability of an eventually negative $B$, which is infeasible. So such a candidate equilibrium must involve a component of $E\left[\overline{B}_t \big/ \left(\overline{P}_t Y_t\right)\right]$ that increases at the rate $\beta^{-t}$. But this means that

$$\liminf_{T\to\infty} \beta^T E\left[\frac{\lambda_T}{\overline{P}_T}\cdot\left(B_T^* - \overline{B}_T\right)\right] = \liminf_{T\to\infty} \beta^T E\left[\frac{B_T^*}{\overline{P}_T Y_T}\right] - \kappa\,, \tag{15}$$

where $\kappa$ is a positive constant because of the exponential growth component of any equilibrium solution of (9). Is it feasible for an individual to choose $B$ so as to make the right hand side of (15) converge to a negative limit? Certainly. In fact it is feasible simply to set $B_T^* = 0$ for all $T$, paying taxes and consuming entirely out of current income every period.

However, the fact that transversality is therefore not satisfied in this type of candidate equilibrium does not prove that these cannot be equilibria – our transversality condition is only sufficient, not necessary. To prove this is not an equilibrium we make a direct argument. The candidate equilibrium requires that $B/PY$ grow arbitrarily large with non-zero probability. The utility function is unbounded. The individual always has the option of consuming his or her entire stock of wealth in the current period, and as $B/PY$ grows larger, the current-period utility increase from doing so grows arbitrarily large. It will then be necessary for the individual to consume at a level less than $C = Y$ thereafter. However, it will always be feasible, for example, to simply set $C = Y - \tau$, never accumulating government debt again, for all periods after that in which the individual's stock of government bonds was liquidated. While the associated expected future utility may be low, even negative, it will be finite and constant. At any date, the expected future utility from sticking with the candidate equilibrium consumption plan, which delivers $C = Y$ forever, is also finite and constant. Thus for large enough $B/PY$, the current utility gain from consuming the entire stock of government debt must appear to the individual to exceed the loss in expected future utility from reverting to the $C = Y - \tau$ policy.

The government could fix $P$ instead of fixing $R$. However in this case it could not at the same time maintain a fixed value of $\tau$. With $P$ and $\tau$ both fixed, the argument that the only value of $B/PY$ consistent with equilibrium is $\xi$ goes through as before. Then we can write the government budget constraint as

$$\xi = R_{t-1}\xi\frac{Y_{t-1}}{Y_t} - \frac{\tau}{Y_t}\;. \tag{16}$$

But this is an equation in which everything is known at time $t-1$ except $Y_t$, which is random. The right-hand side cannot, therefore, always be equal to the left (except in the special case when $\tau = \xi = 0$). So there is no equilibrium with this combination of policies when debt is non-zero.

For there to be an equilibrium when $P$ is pegged, fiscal policy must respond to the level of the debt, for example according to

$$\tau_t = -\phi_0 + \phi_1 \frac{B_t}{P_t} \,. \tag{17}$$

Substituting this policy rule into the government budget constraint, written in real terms, produces, after taking $E_{t-1}$,

$$E_{t-1}\left[\frac{B_t}{P_t Y_t}\right] = \beta^{-1}\frac{B_{t-1}}{P_{t-1}Y_{t-1}} - \left(-\phi_0 E\left[Y^{-1}\right] + \phi_1 E_{t-1}\left[\frac{B_t}{P_t Y_t}\right]\right) \tag{18}$$

which is easily solved to yield a stable difference equation, so long as $(1+\phi_1)\beta > 1$. Thus a large enough $\phi_1$ – one big enough that the increased interest flow from the increase in $B/P$ is fully covered by increased taxes – will prevent the fixed-*P* policy from conflicting with transversality.

Leeper suggests that a "passive" fiscal policy like (17) can be combined with an "active" interest rate policy to produce a unique price level. An active interest rate policy is one that responds strongly to inflation, for example by responding to the price level with a positive coefficient or to the rate of inflation with a coefficient bigger than one. Consider the active interest rate rule

$$R_t = \theta_0 + \theta_1 P_t Y_t \,, \tag{19}$$

combined with the fiscal rule (17). (A rule that makes *R* respond directly to *P* could also be used here, but is messier to analyze. It may even be more realistic to make monetary policy respond to nominal income, rather than *P* itself.) The derivation of (18) is unaffected by the switch from fixed-*P* policy to (19), so it remains true that expected real government debt does not explode. The first-order condition can now be written as

$$\frac{1}{P_t C_t} = (\theta_0 + \theta_1 P_t Y_t)\beta E_t\left[\frac{1}{P_{t+1}C_{t+1}}\right]. \tag{20}$$

Using the fact that $C = Y$ in equilibrium, this becomes an unstable difference equation in $PY$. If $\beta \cdot (\theta_0 + \theta_1 P_t Y_t) > 1$, $E_t\left[1/(P_{t+1}Y_{t+1})\right] < 1/P_t Y_t$, and there is therefore a non-zero probability of $P_{t+1}Y_{t+1} > P_t Y_t$ by some factor greater than one. If this occurs, there is then a non-zero probability of growth at a still higher rate in the next period, and so on. So in this case $P_t Y_t$ grows arbitrarily large with non-zero probability. Similarly, if $\beta \cdot (\theta_0 + \theta_1 P_t Y_t) < 1$, there is non-zero probability of $P_t Y_t$ shrinking arbitrarily close to zero. However, these facts raise no problems for the equilibrium. The only tell us that prices are likely to explode upwards or to implode toward zero. Because of the fiscal policy, real debt will not behave wildly even if *P* does, so the inflation or deflation does not violate transversality or any feasibility condition. Thus an active interest rate policy is not enough to deliver a unique price level in the face of passive fiscal policy. This is similar to what occurs in a model with money, but with the money inessential, so that barter equilibrium is possible. There, too, an active interest rate policy may not guarantee a unique equilibrium with passive fiscal policy. When money is essential, however, it generates a transversality condition and a first order condition that make this type of policy produce a unique price level.