Decentralizing the Growth Model

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I. The Single-Agent-Type Model

The agent maximizes

$$E\left[\sum_{t=0}^{\infty} U(C_t)\right] \tag{1}$$

subject to

$$C_t + I_t = f(K_{t-1}, L_t, A_t)\,,\ t=1,\ldots,\infty\,, \tag{2}$$

$$K_t = I_t + \delta K_{t-1} \tag{3}$$

$$L_t \le 1 \tag{4}$$

and

$$K_t \ge 0\,,\ \text{all } t. \tag{5}$$

We are following the usual (in these notes, not necessarily the world at large) conventions that C, L, K, and I are choice variables and that variables dated t or earlier are known when variables dated t are chosen. Using the methods we have applied earlier on problems like this, we arrive at first-order conditions

∂C: $$U'(C_t) = \lambda_t \tag{6}$$

∂K: $$\lambda_t = \beta E_t\left[\lambda_{t+1}\left(\delta + D_1 f\left(K_t, L_{t+1}, A_{t+1}\right)\right)\right]\,. \tag{7}$$

To facilitate comparison with versions of the model we will take up below, we derive here the linearization of this model about its deterministic steady state. We use (3), (4) and (6) to reduce the system to one in the two variables C and K alone. We also use the fact that, from (7), in steady state

$$\beta^{-1} = f' + \delta\ . \tag{8}$$

Here we are introducing the convention that unsubscripted variables are evaluated at their steady-state values and that $f_t' = D_1 f(K_{t-1}, L_t, A_t)$. We will assume throughout also that f is linear homogenous in K and L jointly, and that A_t enters f as a multiplicative factor. Since we have discussed linearization previously in a more complicated model, here we simply display the linearization:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} dC_t \\ dK_t \end{bmatrix} = \begin{bmatrix} 0 & \beta^{-1} \\ 1 & \dfrac{-f''U'\beta}{U''} \end{bmatrix} \begin{bmatrix} dC_{t-1} \\ dK_{t-1} \end{bmatrix} + \begin{bmatrix} C+K-\delta K \\ -\dfrac{U'}{U''}(1-\beta\delta) \end{bmatrix} dA_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\eta_t \ . \tag{9}$$

The "*d*" operator denotes deviation from steady state. It is not hard to verify that this system has two real roots whose product is β^{-1}.

We will be comparing this solution to other solutions as we go along. Since in some cases analytic solution is clumsy, we will compare two particular special cases numerically. One of these is the Cobb-Douglas, 100% depreciation, log utility case for which we know that an analytic solution exists, with C/K constant. The other will set δ=.9 (10% per period depreciation), but retain the Cobb-Douglas, log utility specification. In both versions of the model our numerical calculations will assume β=.95, α=.3, i.i.d. A_t with mean $\bar{A}=1$. The stable solution for the system has the form

$$\begin{bmatrix} dC_t \\ dK_t \end{bmatrix} = G_1 \begin{bmatrix} dC_{t-1} \\ dK_{t-1} \end{bmatrix} + HdA_t \ . \tag{10}$$

For our two cases, (case 1 is δ=0), we have

Case 1: $$G_1 = \begin{bmatrix} 0 & .7526 \\ 0 & .3 \end{bmatrix},\ H = \begin{bmatrix} .4175 \\ .1664 \end{bmatrix} \tag{11}$$

Case 2: $$G_1 = \begin{bmatrix} 0 & .2146 \\ 0 & .8380 \end{bmatrix},\ H = \begin{bmatrix} .2724 \\ 1.0635 \end{bmatrix}. \tag{12}$$

II. Arrow-Debreu Equilibrium

Now we postulate the existence of a representative firm as well as a representative consumer. The decisions of the firm and consumer are coordinated by market prices, which both types of agent take as unaffected by their decisions. The prices are quoted at time 0, and are contingent both on the date *t* and on the state of the world ω. We think of ω as a point in the set Ω and as determined by an infinite sequence of random variables $\{X_t\}_{t=0}^{\infty}$. Information available at *t* is given by $\omega_t = \{X_s\}_{s=0}^{t}$, which can be thought of as determining a subset of Ω consisting of *X* sequences with the same first *t*+1 elements. The consumer's objective function is still (1), but her budget constraint is now

$$\sum_{\omega\in\Omega} \sum_{t=0}^{\infty} P(t,\omega)\big(C_t(\omega_t) - W_t(\omega_t)L_t(\omega_t) - y_t(\omega_t)\big) = 0\,, \tag{13}$$

where *P* is the goods price, *W* is the wage in goods units, and *y* is the profit distribution by the representative firm, which is owned by the consumer. To justify the use of the summation sign in (13), we have to think of the number of distinct *X* sequences ω as finite, or at least countable, which implies that in finite time uncertainty vanishes. That is, there is some T such that for *t*>T, there is only one ω that matches ω_t. To eliminate this unappealing implication we would have to allow for an uncountably infinite Ω, treat $P(t,\cdot)$ as a function over it, and replace the sum over ω by integrals. This would add mathematical complications, however, so we stick with the assumption of countable Ω for now.

The firm is instructed by its owners to maximize the value of its profit distributions, i.e.

$$\sum_{\omega}\sum_{t=0}^{\infty} P(t;\omega)\, y_t(\omega_t) \ . \tag{14}$$

The firm's constraints, indexed by t and ω_t are

$$y_t(\omega_t) = f(K_{t-1}(\omega_{t-1}), L_t(\omega_t), A_t(\omega_t)) - K_t(\omega_t) + \delta K_{t-1}(\omega_{t-1}) - W_t(\omega_t) L_t(\omega_t) \ . \tag{15}$$

We suppose that there is a probability function π defined on Ω, so that (1) can be rewritten as

$$\sum_{\omega\in\Omega}\pi(\omega)\sum_{t=0}^{\infty}\beta^t U\big(C_t(\omega_t)\big) \ . \tag{16}$$

To make the first-order conditions emerge in a simple form, we introduce the notation

$$p(t;\omega_t) = \beta^{-t}\,\frac{\sum_{\omega\in\omega_t} P(t;\omega)}{\pi(\omega_t)} \ . \tag{17}$$

To justify (17), we have to assume that $\pi_t(\omega_t)$, its denominator, never vanishes when the numerator is nonzero. This is the only restriction we need on π. It does not have to be the "true" probability measure, and it is not necessary that everyone agree that it expresses their beliefs. The equilibrium can be described in terms of the P's without introducing π's at all.

The market-clearing condition in labor is already imposed implicitly through the use of the same symbol L in both the firm and consumer problems. For goods, the market clearing condition is that goods used for consumption and investment at t must match goods produced at t. That is,

$$C_t(\omega_t) + I_t(\omega_t) = f\big(K_{t-1}, L_t, A_t\big) \ . \tag{18}$$

This is not perceived as a constraint by the typical consumer or firm. Each consumer sees her budget constraint (13) as allowing income at any date to be converted into consumption at any date at a rate of tradeoff determined by the $P(t;\omega)$ values. But the prices must adjust so that in equilibrium consumers choose to consume an amount consistent with the amounts of output and investment chosen by producers.

First-order conditions for the consumer are

∂C_t: $$\beta^t \pi(\omega_t) U_t' = \lambda \beta^t \pi(\omega_t) p(t;\omega_t),\ \therefore U_t' = \lambda p(t;\omega_t) \tag{19}$$

First-order conditions for the firm are

∂y_t: $$p(t;\omega_t) = \mu_t(\omega_t) \tag{20}$$

∂L_t: $$\mu_t(\omega_t)\big(D_L f_t - W_t\big) = 0 \tag{21}$$

$$\partial K_t: \quad \begin{aligned} \pi(\omega_t)\mu(t;\omega_t) &= \sum_{\omega_{t+1}\in\omega_t} \pi(\omega_{t+1})\mu(t+1;\omega_{t+1})\left(\delta + D_K f_{t+1}\right) \\ &= \pi(\omega_t) E_t\left[\mu(t+1)\left(\delta + D_K f_{t+1}\right)\right] \end{aligned} . \tag{22}$$

In deriving (20)-(22) we are using the convention that in the Lagrangian the multiplier on the t, ω_t constraint of the firm is $\beta^t \pi(\omega_t)\mu(t;\omega_t)$. Also, we are using the fact that $\pi(\omega_{t+1})/\pi(\omega_t)$ is the conditional probability of ω_{t+1} given that ω_t has occurred.

The FOC's (6) and (7) from the single-agent problem can be solved to eliminate λ, and the resulting equation can also be derived from (19), (20), and (22). Thus an Arrow-Debreu equilibrium satisfies the constraints and FOC's of the single-agent problem.

III. Autonomous Firms

We now consider the opposite extreme case, in which the representative firm gets no guidance from asset prices in making investment decisions. We postulate that the firm's objective function is increasing in its profit distributions, but that it has its own "utility function" for those distributions that does not have any necessary link to the representative consumer's utility function. That is, we postulate that firms maximize

$$E\left[\sum_{t=0}^{\infty} \theta^t \phi(y_t)\right], \tag{23}$$

with $\phi' > 0$, $\phi'' \leq 0$. The firm's discount factor θ is allowed to differ from that of consumers. A version of this assumption that appears regularly in the applied literature on investment is that (23) holds with $\theta = \beta$ and ϕ the identity function, so that firms maximize expected current and future profits discounted at the fixed rate β. The firm's constraints are

$$\left.\begin{aligned} y_t &= f(K_{t-1}, L_t, A_t) - K_t + \delta K_{t-1} - W_t L_t \\ K_t &\geq 0 \end{aligned}\right\}, \text{ all } t\,. \tag{24}$$

The consumer still maximizes the objective function (1), but now with the constraints

$$\begin{aligned} C_t &= W_t L_t + y_t \\ L_t &\leq 1 \end{aligned}, \text{ all t.} \tag{25}$$

Note that the consumer here has no intertemporal decision to make at all. Because we have assumed leisure has no utility, L_t will always be at its upper bound of 1. Individuals take wages W and profit distributions y as beyond their control, so (25) determines C without any reference to the consumer's objective function.

Obviously here the only interesting economic decision in the economy, the choice of how much to invest and how much to consume each period, is being made by the firm, using an objective function that does not match that of the people in the economy. We should not expect the resulting equilibrium to be close to optimal in general. It is interesting to ask, though, whether with the firm's discount factor θ matching the individual's discount factor β, there might be ϕ's for which the solution to this problem would match or come close to the social optimum. The firm's FOC's are

∂y: $$\phi'(y_t) = \mu_t \tag{26}$$

∂L: $$W_t = D_L f(K_{t-1}, L_t, A_t) \tag{27}$$

∂K: $$\mu_t = \theta E_t\left[\mu_{t+1}\left(D_K f(K_t, L_{t+1}, A_{t+1}) + \delta\right)\right] \tag{28}$$

where μ_t is the Lagrange multiplier on the constraint. Note that in the deterministic steady state we will have

$$\theta \cdot \left(D_K f(\overline{K},1,1) + \delta\right) = 1 \ . \tag{29}$$

By comparing (29) to (7), we see that the values of $\overline{K}$, and therefore $\overline{C}$, match those of the single-agent model if $\theta = \beta$, but not otherwise.

When, as in our case 1, $\delta = 0$, and in addition ϕ is logarithmic and $\theta = \beta$, this autonomous-firm model can be shown to have an exact solution in which y remains proportional to K, and this gives the same C time paths as the complete markets solution. However outside this special case, as in our case 2, the model does not have an analytic solution. To study its behavior we linearize. Here we reduce the model to a two-variable system in y and K, using (27) to eliminate W, arriving at

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} dy_t \\ dK_t \end{bmatrix} = \begin{bmatrix} 0 & \theta^{-1} + Kf'' \\ 1 & -f''\theta\dfrac{\phi'}{\phi''} \end{bmatrix}\begin{bmatrix} dy_{t-1} \\ dK_{t-1} \end{bmatrix} + \begin{bmatrix} \left(y + K - \delta K\right)/A \\ -\dfrac{\phi'}{\phi'' A}(1-\theta\delta) \end{bmatrix} dA_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\eta_t \ . \tag{30}$$

To give this system the greatest possible chance of matching the complete markets solution, we will examine it numerically for the case where ϕ is the log function and $\theta=\beta$, with the rest of the parameters set as in cases 1 and 2 of section I. Note that with the matching discount rates, the steady states of this model do match those of the complete markets model. This leads to G_1 and H matrices as follows:

Case 1: $$G_1 = \begin{bmatrix} 0 & .0158 \\ 0 & .3 \end{bmatrix}, \ H = \begin{bmatrix} .0088 \\ .1664 \end{bmatrix}. \tag{31}$$

Case 2: $$G_1 = \begin{bmatrix} 0 & .0473 \\ 0 & .8985 \end{bmatrix}, \ H = \begin{bmatrix} .02 \\ .3807 \end{bmatrix}. \tag{32}$$

Because this system is in terms of y and K instead of C and K, the matrices displayed here would not match those of section I even if C and K followed the same paths. However, since the second row both here and in section I is a difference equation in K alone, we can see immediately whether the solutions match, for either case. If both the second row of G_1 and the lower element of H match, then the solutions are first-order equivalent, as the K paths will be and the social resource constraint determines the C path from the K path. We can see that for case 1 the autonomous firm solution does match the complete markets solution to first order, as we would hope given that we know this is true of the exact solution. However in case 2 the solutions do not match. The coefficient on lagged K is slightly larger in the autonomous-firm model, but more importantly the responsiveness of K to shocks in A is several times bigger in the complete-markets model. The firm tends to smooth K's time path more than is optimal, letting shocks have larger current impact on C.

IV. Incomplete Asset Markets

As you should know from your micro theory course, the real allocations in an Arrow-Debreu equilibrium can generally be duplicated without a complete market for claims arbitrarily far in the future, if instead there is a one-period-ahead complete contingent claims market at each date. In a model like the stochastic growth model, where the disturbance A_t is generally thought of as continuously distributed,[1] this would require an uncountable infinity of assets to be traded at each date. Since in reality a finite number of assets are traded, it is interesting to explore how a competitive equilibrium with a few assets compares to one where there are complete markets.

We consider the special case where there is a single traded asset, denominated in shares, with the return at t per share purchased at time t-1 denoted z_t. We do not yet take a position on what the stochastic process of z will be, though we will assume that it is regarded by agents as unaffected by individual agents' actions. We denote by Q_t the price of the asset at time t. Because consumers can no longer instruct firms to maximize the value of the stream of profits (because there are no quoted prices to use to value this uncertain stream), there must be some other objective function given the firm.

The consumers' objective function is still (1), but her budget constraint becomes

$$C_t + Q_t S_t = W_t L_t + \left(Q_t + z_t\right) S_{t-1} + y_t \ . \tag{33}$$

We need also to impose a constraint, which we hope does not bind in equilibrium, that indefinite borrowing is not possible, e.g. that

$$Q_t S_t \geq -B\nu^t \tag{34}$$

for some constants B>0 and $\nu \in (0, \beta^{-1})$.

The firm still has the objective function (23), but now with the constraint

$$y_t - Q_t S_t = f(K_{t-1}, L_t, A_t) - K_t + \delta K_{t-1} - W_t L_t - \left(Q_t + z_t\right) S_{t-1} \ . \tag{35}$$

Note that we use the same symbol S for both the purchases of the security by the consumers and the sales of the security by the firm, implicitly imposing market clearing with zero net supply of the security. The firm also must have a limit on its borrowing, which becomes (since S is securities issued by the firm)

$$Q_t S_t \leq B\nu^t \ . \tag{36}$$

Now the consumer's FOC's are (6) and

[1] That is, with a density function on the real line, like the log-normal or exponential for example.

$$\lambda_t Q_t = \beta E_t\left[\lambda_{t+1}\cdot\left(Q_{t+1}+z_{t+1}\right)\right]. \tag{37}$$

The firm's FOC's are

∂y_t:
$$\phi'(y_t)=\mu_t \tag{38}$$

∂L_t:
$$\mu_t\cdot\left(W_t - D_L f_t\right)=0 \tag{39}$$

∂K_t:
$$\mu_t = \theta E_t\left[\mu_{t+1}\cdot\left(\delta + D_K f_{t+1}\right)\right] \tag{40}$$

∂S_t:
$$\mu_t Q_t = \theta E_t\left[\mu_{t+1}\cdot\left(Q_{t+1}+z_{t+1}\right)\right]. \tag{41}$$

This is more promising than the autonomous firm model. It is still true that (40) uses μ instead of λ as the stochastic discount factor, but now we have (41) and (37), which seem to require at least some similarity in behavior between the discount factors $\theta\mu_{t+1}/\mu_t$ and $\beta\lambda_{t+1}/\lambda_t$. From (6) we know that in steady state λ is constant. Then from (37) we conclude that Q is also constant, at the value $z/(\beta^{-1}-1)$. Then (41) implies

$$\mu_{t+1}/\mu_t = \beta/\theta\ . \tag{42}$$

Using this in (40) lets us conclude that in this model we have the same equation (7) determining steady state capital stock as in the single agent model, even when $\beta\neq\theta$.

Suppose we linearize the system about steady state. We now have to keep track of more variables, because *y*, *C*, *K*, *Q* and *S* all interact. We also stick to the case $\beta=\theta$ to avoid having to deal with trends in *y* and *S* in "steady state." We write the system as

$$\Gamma_0 dx_t = \Gamma_1 dx_{t-1} + \Psi d\varepsilon_t + \Pi\eta_t\ , \tag{43}$$

where the η vector are endogenous prediction errors and the ε vector is exogenous disturbances. We define

$$x = \begin{bmatrix} y \\ Q \\ S \\ K \\ C \end{bmatrix},\ \varepsilon = \begin{bmatrix} A \\ z \end{bmatrix} \tag{44}$$

and arrange the equations in the order (35), (35)+(33) (the social resource constraint), (37), (40), and (41). The result is

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & -Q & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & \beta & 0 & 0 & -Q\gamma_U \\ -\gamma_\phi & 0 & 0 & 0 & 0 \\ -\gamma_\phi Q & \beta & 0 & 0 & 0 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} 0 & 0 & -(Q+1) & \beta^{-1}+Kf'' & 0 \\ 0 & 0 & 0 & \beta^{-1} & 0 \\ 0 & 1 & 0 & 0 & -\gamma_U Q \\ -\gamma_\phi & 0 & 0 & -f''\beta & 0 \\ -\gamma_\phi Q & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{45}$$

where $\gamma_\phi = -\phi''/\phi'$ and $\gamma_U = -U''/U'$. We also have

$$\Psi = \begin{bmatrix} -S & \alpha f \\ 0 & f \\ -\beta & 0 \\ 0 & -\beta f' \\ -\beta & 0 \end{bmatrix}, \Pi = \underset{5\times 3}{\begin{bmatrix} 0 \\ I \end{bmatrix}}. \tag{46}$$

To keep our numerical solutions comparable, we will take the steady state to have S=0. The solutions for this model in our two cases are

Case 1
$$G_1 = \begin{bmatrix} 0 & 0 & -1 & .0158 & 0 \\ 0 & 0 & 0 & 33.532 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & .7526 & 0 \end{bmatrix}, H = \begin{bmatrix} .0088 \\ 18.6014 \\ 0 \\ .1664 \\ .4175 \end{bmatrix} \tag{47}$$

Case 2
$$G_1 = \begin{bmatrix} 0 & 0 & -1 & .0473 & 0 \\ 0 & 0 & 0 & 3.0186 & 0 \\ 0 & 0 & 1 & -.0032 & 0 \\ 0 & 0 & 0 & .8380 & 0 \\ 0 & 0 & 0 & .2146 & 0 \end{bmatrix}, H = \begin{bmatrix} .02 \\ 3.8309 \\ .0359 \\ 1.0635 \\ .2724 \end{bmatrix} \tag{48}$$

Comparing (47) with (11) and (48) with (12), we see that the last two rows of the larger system match the smaller system exactly. This system, like that with autonomous firms, exactly reproduces the complete-markets equilibrium in case 1. Both y and C in that case move exactly in proportion to K, so that ϕ' and U' move exactly in proportion. This is signaled in the linearized solution by the fact that the third element of H, the effect of a disturbance in A on S, is zero. That is, the linearization implies that if we start with S=0, random shocks do not generate non-zero S. Without this condition, y and C could not remain proportional, as the third row of G_1 implies that S, once perturbed away from zero, will tend to drift. (The unit coefficient on lagged S implies that S has no tendency to return to steady state.) The first row of G_1 implies that y will tend to follow (with opposite sign) any drift in S. In case 2, we see from the third element of H that random disturbances do affect S, so for this case S and y will drift away from their steady state values and y and C do not remain strictly proportionate.

It is therefore not possible in case 2 that the random discount factor that firms use to evaluate investments, $\beta\phi'_{t+1}/\phi'_t = \beta y_t/y_{t+1}$, is exactly the same as what consumers would use, i.e. $\beta U'_{t+1}/U'_t = \beta C_t/C_{t+1}$. There will be differences in the time paths of K and C between the complete and incomplete markets economies. The match between the linearized solutions implies, though, that the differences will be small as a proportion of variation in the economy when the stochastic disturbances to the economy are small.