Answers to Dynamic Programming Exercise

Note: These answers are more complete and careful than was expected as standard performance on the exercise.

1. This problem is already in the standard dynamic programming form, with W as state, C as control, and the i.i.d. random variable Y as disturbance. The Bellman equation is therefore

$$V(W_t) = C_t - \tfrac{1}{2}C_t^2 + \mathrm{q}\left(W_t - \tfrac{1}{2}W_t^2\right) + \mathrm{b}E_t\left[V\left(R\cdot(W_t - C_t) + Y_{t+1}\right)\right] , \qquad \text{(i)}$$

where $R = 1 + r$. The first order conditions are then

$$1 - C_t = R\mathrm{b}E_t V'_{t+1} . \qquad \text{(ii)}$$

$$V'_t = \mathrm{q}(1 - W_t) + R\mathrm{b}E_t V'_{t+1} . \qquad \text{(iii)}$$

Using the condition given in the problem statement that $R\mathrm{b} = 1$, and solving to eliminate V'_t, we get

$$V'_t = \mathrm{q}\cdot(1 - W_t) + 1 - C_t \qquad \text{(iv)}$$

$$1 - C_t = E_t\left[\mathrm{q}\cdot(1 - W_{t+1}) + 1 - C_{t+1}\right] . \qquad \text{(v)}$$

Using the assumption that in this linear-quadratic problem we will have a linear decision rule $C_t^* = a + bW_t$, we obtain

$$\begin{aligned} 1 - a - bW_t &= E_t\left[\mathrm{q}\cdot(1 - W_{t+1}) + 1 - a - bW_{t+1}\right] \\ &= \mathrm{q} + 1 - a - (\mathrm{q} + b)\left(R\cdot(W_t - a - bW_t) + \bar{Y}\right) \\ &= \mathrm{q} + 1 - a - (\mathrm{q} + b)\left(\bar{Y} - Ra\right) - (\mathrm{q} + b)R(1 - b)W_t \end{aligned} \quad . \qquad \text{(vi)}$$

Equating coefficients on the two sides of (vi) gives us

$$\begin{aligned} & b = (\mathrm{q} + b)R\cdot(1 - b), \therefore b = \frac{1 - \mathrm{q} - R^{-1} \pm \sqrt{\left(1 - \mathrm{q} - R^{-1}\right)^2 + 4\mathrm{q}}}{2} \\ & 0 = \mathrm{q} - (\mathrm{q} + b)\left(\bar{Y} - Ra\right), \therefore a = R^{-1}\cdot\left(\bar{Y} - \frac{\mathrm{q}}{\mathrm{q} + b}\right) \end{aligned} \quad . \qquad \text{(vii)}$$

We can see from the first equation in (vii) that the two solutions for b are both real and have opposite sign for any $\mathrm{q} > 0$.

In order for W (and hence C) to be a martingale, substituting the decision rule in for C in (2) must make the coefficient on lagged W in the resulting equation 1. This requires $R\cdot(1 - b) = 1, \therefore b = 1 - R^{-1}$, and it can be verified that this occurs only when $\mathrm{q} = 0$.

The value function, obtained by substituting our linear decision rule in (iv) and integrating with respect to W, is

$$V(W) = \int \left(\theta \cdot (1 - W) + 1 - a - bW\right) dW = \kappa + (\theta + 1 - a)W - \frac{(\theta + b)}{2} W^2 . \qquad \text{(viii)}$$

Recall that it is a necessary condition for an optimum requiring that $E\left[\beta^t V(W_t)\right]$ have a limit of zero on an optimally chosen path for W. Then, since (viii) is quadratic in W, clearly we must have $E\left[\beta^t W_t^2\right] \to 0$ on such paths. Since equation (2) is a univariate difference equation with root $R \cdot (1 - b)$, and $R\beta = 1$, we require $R \cdot (1 - b)^2 < 1$ to meet this condition. this condition is not met when $b \le 0$, because we are assuming $R\beta = 1$, which implies $R > \beta^{-\frac{1}{2}}$ (as long as β<1). Thus only the positive root for b is a potential solution. That the condition $E\left[\beta^t W_t^2\right] \to 0$ holds for the positive solution to (vii), because $R \cdot (1 - b)^2 < 1$ for that root, is hard to prove directly from the formula for the root given in (vii). However it can be verified numerically and proved with a tedious argument based on evaluating derivatives of $R \cdot (1 - b)^2$ with respect to θ and R. (The limiting value for the expression as $R \to 1$ and $\theta \to 0$ is 1, and the derivatives of the expression, with respect to both θ and R are negative over the entire region $R > 1$, $\theta > 0$.)

As to the differences in the solution to this problem vs. the problem we studied in class, first note that for this problem we did not ever invoke the $E\left[\beta^t W_t^2\right] \to 0$ as a constraint. It emerges as an implication of one of the necessary conditions for a maximum. It is easy to check that the solution to this problem converges to that for the classroom problem as $\theta \to 0$. Thus one can justify the apparently strange constraint $E\left[\beta^t W_t^2\right] \to 0$ as a way of generating the limiting solution to a problem with some very small weight on deviation of W from an optimum level. But away from this limiting case, this problem always produces a stationary solution for W, and hence C, despite the fact that we have $R\beta = 1$, whereas the classroom problem produced martingale behavior for both W and C.

Since the $E\left[\beta^t W_t^2\right] \to 0$ constraint is redundant in this problem, we could drop it altogether without changing the solution. This is a big contrast from with the classroom problem. However adding the constraint $W_t \ge C_t$ might or might not alter the solution. (Here, as in class, I mistakenly gave the constraint as $W_t \ge 0$, which generally makes no solution exist, unless there are constraints on the distribution of Y.) The question is whether , $W_t \ge C_t$ too, would always be non-binding when C is chosen according to the optimal decision rule, $W_t - C_t = (1 - b)W_t - a$. The question then is whether, for every possible W, i.e. for every W>0, this expression is positive. This will be true if and only if $a < 0$. We can see from the expression for a in (vii) that this depends on $\bar{Y}$, θ, and R. We can see that if $\bar{Y} > 1$, the condition could never be met. For small $\bar{Y}$, the condition will be met if θ is not too small. Intuitively, the problem is that with $\bar{Y}$ large, optimal behavior may involve borrowing against future income.

2. There is a typo in equation (5) on the problem statement. It is convenient if $\overline{Y}$ represents the mean of the stationary AR(1) process Y_t, so equation (5) should read $Y_t = \alpha Y_{t-1} + (1-\alpha)\overline{Y} + \varepsilon_t$. The problem is still internally consistent with the original version of (5), but the mean of Y is then $\overline{Y} \cdot (1-2\alpha)/(1-\alpha)$.

The main difference between this problem and the one we did in class is that now there are two state variables W_t and Y_t, with:

$$W_{t+1} = f(W_t, Y_t, C_t) \equiv (1+r)(W_t - C_t) + \alpha Y_t + (1-\alpha)\overline{Y} + \varepsilon_{t+1}; \tag{ix}$$

and

$$Y_{t+1} = g(W_t, Y_t, C_t) \equiv \alpha Y_t + (1-\alpha)\overline{Y} + \varepsilon_{t+1}. \tag{x}$$

The intertemporal optimization problem is to maximize (4) subject to (ix), (x) and the "wealth growth" constraint (3), with W_0 and Y_0 given. The corresponding Bellman equation is given by:

$$V(W_t, Y_t) = \max_{C_t} \left\{ (C_t - \tfrac{1}{2} C_t^2) + \beta E_t \left[V(W_{t+1}, Y_{t+1}) \right] \right\} \tag{xi}$$

from which we obtain the first order conditions:

$\partial/\partial C_t$: $$1 - C_t = \beta(1+r) E_t V_{1t+1} \tag{xii}$$

$\partial/\partial W_t$: $$V_{1t} = \beta(1+r) E_t V_{1t+1} \tag{xiii}$$

where V_{1s} refers to the partial derivative of $V(W_s, Y_s)$ with respect to its *first* argument, W_s, at time s. [While another envelope condition can be written for $\partial/\partial Y_t$, it is not needed to find the solution].

Since this is a linear-quadratic problem, we know that the optimal policy function $C^*(W_t, Y_t)$ will be linear in W_t and Y_t. Therefore, we "guess" that $C_t^* = a + bW_t + cY_t$, plug in this guess into our equations and perform algebraic manipulations to determine a, b and c. For any candidate solution that is thus found, we then need to check that the "wealth growth" constraint (3) is not violated.

Proceeding in this manner, we have from (xii) and (xiii):

$$V_{1t} = 1 - C_t \quad \Rightarrow \quad 1 - C_t = \beta(1+r) E_t \left[1 - C_{t+1} \right] \tag{xiv}$$

and then plugging in our guess for C_t, we have:

$$\begin{aligned} 1 - a - bW_t - cY_t &= \beta(1+r) E_t \left[1 - a - bW_{t+1} - cY_{t+1} \right] \\ &= \beta(1+r) E_t \Big[1 - a - b\left[(1+r)(W_t - C_t) + \alpha Y_t + (1-\alpha)\overline{Y} + \varepsilon_{t+1} \right] \\ &\qquad - c\left[\alpha Y_t + (1-\alpha)\overline{Y} + \varepsilon_{t+1} \right] \Big] \end{aligned}$$

$$= \beta(1+r)E_t\Big[1-a-b\big[(1+r)(W_t-a-bW_t-cY_t)+\alpha Y_t+(1-\alpha)\overline{Y}+\epsilon_{t+1}\big]$$
$$-c\big[\alpha Y_t+(1-\alpha)\overline{Y}+\epsilon_{t+1}\big]\Big]$$

Equating the constant, W_t and Y_t coefficient terms on the left-hand and right-hand sides, we get the following three equations in *a*, *b* and *c*:

$$1-a = \beta(1+r)\big[1-a+b(1+r)a-b(1-\alpha)\overline{Y}-c(1-\alpha)\overline{Y}\big] \quad \text{(xv)}$$

$$b = \beta(1+r)\big[b(1+r)(1-b)\big] \quad \text{(xvi)}$$

$$c = \beta(1+r)\big[b\alpha+c\alpha-b(1+r)c\big] \quad \text{(xvii)}$$

In solving (xv)-(xvii) for *a*,*b* and *c*, we assume that $r>0,\ 1>\alpha>0$ and $1>\beta>0$. Depending on whether $b=0$ or $b\neq 0$, two candidate solution classes can be determined.

If $b=0$, equation (xvi) is satisfied trivially. Plugging $b=0$ into (xvii) and solving for *c*, we get $c=0$; and plugging $b=0,\ c=0$ into (xv), we get $a=1$. This then gives us the candidate solution $C_t=1,$ for all t. We note that if $1+r<\beta^{-1/2}$, this candidate solution does not violate condition (3), since with $C_t=1$, W_t grows at the rate of $1+r$, which implies that $\beta^{t/2}W_t$ grows at the rate $\beta^{1/2}(1+r)$ which by assumption is less than one. Similarly, if $1+r\geq\beta^{-1/2}$, then the growth rate of $\beta^{1/2}(1+r)$ is greater than or equal to one, and condition (3) is violated. We note that this solution of $C_t=1$ for $1+r<\beta^{-1/2}$ corresponds to achieving the satiation level of consumption at all points in time.

If $b\neq 0$, we can solve for *b* from equation (xvi) to get $1-b=\dfrac{1}{\beta(1+r)^2}$; and then successively solve for *c* from equation (xvii) and *a* from equation (xv). Checking the "wealth growth" constraint (3), we note that for $b\neq 0$, W_t grows at the rate of $(1+r)(1-b)=\dfrac{1}{\beta(1+r)}$. This implies that $\beta^{t/2}W_t$ grows at the rate $\left[\beta^{1/2}(1+r)\right]^{-1}$.

Thus, for $b\neq 0$, we have condition (3) violated if $1+r\leq\beta^{-1/2}$ and satisfied if $1+r>\beta^{-1/2}$. Therefore, we have this class of solutions available for the case of $1+r>\beta^{-1/2}$.

We note that for the case of $1+r=\beta^{-1/2}$, neither the $b=0$ nor the $b\neq 0$ candidate solutions noted above are available, since for this case condition (3) is violated in each instance. Therefore, there is no optimal solution for this case.

We also note that the $b\neq 0$, $1+r>\beta^{-1/2}$ solution class can incorporate any of the three possibilities of $\beta(1+r)>1$, $\beta(1+r)=1$ and $\beta(1+r)<1$. For the $\beta(1+r)=1$ case, the computations are fairly straightforward. awe have the usual result from (xiv) that consumption follows a random walk. The solution for *b*, *c* and *a* that are determined successively from equations (xvi), (xvii) and (xv) gives us the optimal policy function for this case of :

$$C_t = \frac{r}{1+r}W_t + \frac{r\alpha}{(1+r)(1+r-\alpha)}Y_t + \frac{(1-\alpha)\bar{Y}}{1+r-\alpha} \tag{xviii}$$

For the case where $\beta \cdot (1+r) \neq 1$, the solution is

$$C_t = \left(1 - \frac{1}{\beta R^2}\right)W_t + \left(\frac{\alpha \cdot \left(\beta R^2 - 1\right)}{\beta R^2 \cdot (R-\alpha)}\right)\left(Y_t - \bar{Y}\right) + \frac{\beta R^2 \cdot \left(\bar{Y} - 1\right) + R \cdot \left(1 + \beta\alpha\right) - \bar{Y} - \alpha}{\beta R \cdot (R-\alpha)(R-1)} \;, \tag{xix}$$

where $R = 1 + r$. For this more general solution, the algebra is tedious, especially for finding the constant term. The constant term part of (xix) was derived for this answer sheet using Maple, a symbolic computation package (like the more widely available Mathematica package usable at Statlab). Note that when $1 + r < \beta^{-\frac{1}{2}}$, according to (xix), consumption decreases when wealth increases, which can easily be seen to imply that wealth explodes faster than the rate implied by the constraint (3).

Finally, we can determine the value functions that correspond to our solutions above.

For the solution $C_t = 1,\ 1 + r < \beta^{-1/2}$ we have $V_0 = \sum_{t=0}^{\infty} (1/2)\beta^{t/2} = \dfrac{1}{2(1-\beta)}$.

For the solution given by $b \neq 0$, $\beta(1+r) = 1$ and optimal policy function (xviii), we have $V_{1t} = 1 - C_t = \left[1 - \dfrac{(1-\alpha)\bar{Y}}{1+r-\alpha}\right] - \dfrac{r}{1+r}W_t - \dfrac{r\alpha}{(1+r)(1+r-\alpha)}Y_t$. Since V_{1t} represents the partial derivative of $V(W_t, Y_t)$ with respect to W_t, we can recover $V(W_t, Y_t)$ by integrating this expression for V_{1t} with respect to W_t. When we do this, we get the solution

$$V(W_t, Y_t) = \left[1 - \frac{(1-\alpha)\bar{Y}}{1+r-\alpha}\right]W_t - \frac{r}{2(1+r)}W_t^2 - \frac{r\alpha}{(1+r)(1+r-\alpha)}Y_t W_t + K\left(Y_t\right), \tag{xx}$$

where K is a "constant of integration" that, since the integration is with respect to W_t, can depend on Y_t. To find the K function, it is helpful to use the envelope condition with respect to Y. (Note that often only a decision rule is needed, and as in this example the envelope condition with respect to the exogenous state is not needed for finding the decision rule.) The Y envelope condition is

$$V_{2,t} = \beta E_t\left[((V_{2,t+1} + V_{1,t+1}) \cdot \alpha\right]. \tag{xxi}$$

From (xx) we can see that

$$V_2\left(W_t, Y_t\right) = -\frac{r\alpha}{(1+r)(1+r-\alpha)}W_t + K'\left(Y_t\right). \tag{xxii}$$

Maintaining our guess of a quadratic form for V, we expect $K'(Y)$ to take a linear form, say $g + hY$. Using (xxii) in (xxi) and equating coefficients on equivalent terms in the polynomials on the two sides of the equation we emerge with

$$g = \mathrm{ba}g + \mathrm{ba}\left[1 - \frac{(1-\mathrm{a})\overline{Y}}{1+r-\mathrm{a}}\right],\ \therefore\ g = \frac{\mathrm{ba}}{1-\mathrm{ba}}\left[1 - \frac{(1-\mathrm{a})\overline{Y}}{1+r-\mathrm{a}}\right]$$

$$h = \mathrm{ba}h + \mathrm{ba} - \frac{r\mathrm{a}}{(1+r)(1+r-\mathrm{a})},\ \therefore h = \frac{\mathrm{a}}{1-\mathrm{ba}}\left(\mathrm{b} - \frac{r}{(1+r)(1+r-\mathrm{a})}\right).$$

A pure constant of integration, which we might call k, will remain. As noted in class, k can be found by substituting into the Bellman equation $V(W_t, Y_t) = C_t^* - \frac{1}{2}C_t^{*2} + \mathrm{b}E_t V(W_{t+1}, Y_{t+1})$ the expressions for $V(W_t, Y_t)$ and $V(W_{t+1}, Y_{t+1})$ as above and the expression for C_t^* as in (xviii). The expression for $V(W_{t+1}, Y_{t+1})$ in the Bellman equation can then be expressed using (xxiii), (x) and again (xviii) in terms of $W_t, Y_t, \overline{Y}$ and e_{t+1}. By passing the expectation operator through and matching coefficients on the constant term on both the left-hand side and the right-hand side of the Bellman equation, we can determine k. Working out this detail is tedious algebra and is omitted here. When we have solved for k in this manner, the value function for initial W_0, Y_0 is given by:

$$V(W_0, Y_0) = \left[1 - \frac{(1-\mathrm{a})\overline{Y}}{1+r-\mathrm{a}}\right]W_0 - \frac{r}{2(1+r)}W_0^{\,2} - \frac{r\mathrm{a}}{(1+r)(1+r-\mathrm{a})}Y_0 W_0 + gY_0 + \frac{h}{2}Y_0^2 + \mathrm{k}\ .$$