Stochastic taxes on nominal bonds

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Stochastic Lagrange Multipliers with expectational constraints

$$\max_{Y} E\left[\sum_{t=0}^{\infty} \beta^t U(y_t, y_{t-1}, z_t)\right]$$

subject to $\quad g(y_t, y_{t-1}, z_t) \leq 0$ for $t = 0, \ldots, \infty$

$$E_t[h(y_{t+1}, y_t, z_{t+1})] \leq 0 \text{ for } t = 0, \ldots, \infty\,.$$

Lagrangian and FOC

Lagrangian

$$\beta^t \big(U(y_t, y_{t-1}, z_t) - \lambda_t g(y_t, y_{t-1}, z_t) - \mu_t E_t[h(y_{t+1}, y_t, z_{t+1})]\big)$$

FOC

$$\begin{aligned} D_1U(y_t, y_{t-1}, z_t) + \beta E_t[D_2U(y_{t+1}, y_t)] \\ = \lambda_t D_1 g(y_t, y_{t-1}, z_t) + \beta E_t[\lambda_{t+1} D_2 g(y_{t+1}, y_t, z_{t+1})] \\ + \mu_t E_t[D_2 h(y_{t+1}, y_t, z_{t+1})] + \beta^{-1}\mu_{t-1} D_1 h(y_t, y_{t-1}, z_t)\,. \end{aligned}$$

Comments

Problems like this often arise when we consider optimal government policy, assuming that the government treats private sector FOC's as constraints. This makes sense when we are looking at a full commitment, rational expectations solution. The government chooses rules that map future exogenous disturbances into future policy choices, the public understands those rules and the stochastic properties of the future xhocks, and the government is committed to following those rules.

But note that there is no $E_{-1}h(y_0, y_{-1}, z_0) \leq 0$ constraint. That is because the problem starts at time 0, so the beliefs of private agents at time $t = -1$ about what would happen at $t = 0$ do not put any constraint on the government optimizer. So the FOC given is only valid for $t \geq 1$. At $t = 0$ the last term, involving μ_{t-1} drops out.

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- Assumes deadweight loss from constant revenues is constant, i.e. something like a labor tax, not a capital tax.

Government's problem

(1)

(2)

(3)

(4)

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Obj.fcn: $$\max_{\tau_s, B_s} E\left[\sum_{t=0}^{\infty} -\ell(\tau_t)\beta^t\right] \quad (1)$$

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Government's problem

$$\text{Obj.fcn:} \quad \max_{\tau_s, B_s} E\left[\sum_{t=0}^{\infty} -\ell(\tau_t)\beta^t\right] \tag{1}$$

$$\lambda: \quad B_t + \tau_t \geq \rho_{t-1}B_{t-1} + G_t \tag{2}$$

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- (3) implies absence of risk aversion, or else that consumption is constant.

- By using a function of τ alone to stand for deadweight loss from taxes and τ to stand for total revenues, the model avoids treating explicitly fluctuations in the tax base and possible effects of tax rates on the tax base.

FOC's

$$\partial\tau: \qquad \ell'(\tau_t) = \lambda_t$$

$$\partial B: \qquad \lambda_t = E_t \lambda_{t+1}$$

- The FOC's deliver immediately the conclusion that $\ell'(\tau_t)$ is a martingale.
- If ℓ'' is everywhere positive, so there is a one-one mapping between ℓ' and τ, this implies that in the absence of uncertainty, τ itself will be set at a constant.
- The government's no-Ponzi constraint (4) together with its budget constraint require that, when τ_t itself is a martingale, the planned

constant level of taxes match the interest rate times current debt plus the discounted present value of future G:

$$\tau_t = (\beta^{-1} - 1)\left(B_t + E_t\left[\sum_{s=1}^{\infty}\beta^s G_{t+s}\right]\right) \tag{5}$$

Uncertainty

- If ℓ is quadratic, then ℓ' is linear and τ itself is implied to be a martingale.

- It is natural to suppose that ℓ must, at least for high values of τ, increase more rapidly than a quadratic, since tax rates and current output are inherently bounded, and one supposes that as tax revenues approach their maximum, the deadweight loss must increase very rapidly.

- So assume that ℓ' is convex, and thus that τ is a concave function of ℓ'.

- $Ex = \bar{x} \Rightarrow Ef(x) \leq f(\bar{x})$ for any concave function f, so $E\tau_{t+1} \leq \tau_t$.

- That is, in the presence of uncertainty expected revenues decline over time. In effect, uncertainty makes it optimal to tax more heavily now as insurance against being driven to very inefficient high tax rates in the future. This is a "second order" effect, though. Unless τ is very large, the optimal expected rate of decrease in τ over time is likely to be small.

- Even in the quadratic case considered by Barro, though expected future τ is constant, actual τ adjusts, period-by-period, to any random disturbances in current and future G. It is not hard to verify that under these conditions B, like τ, is a martingale.

Adding a price level to the Barro model

$$\lambda: \qquad \frac{B_t}{P_t} + \tau_t \geq \rho_{t-1}\frac{B_{t-1}}{P_t} + G_t \tag{6}$$

$$\mu: \qquad 1 \leq \beta\rho_t E_t\left[\frac{P_t}{P_{t+1}}\right] \tag{7}$$

$$\lim_{t\to\infty} E\left[\beta^t \frac{B_t}{P_t}\right] = 0 \text{ alone or } \textit{also} \text{ with } B_t \geq 0 \tag{8}$$

$$\partial\tau: \qquad \ell'(\tau_t) = \lambda_t$$

$$\partial B: \qquad \lambda_t = \beta\rho_t E_t\left[\lambda_{t+1}\frac{P_t}{P_{t+1}}\right]$$

$$\partial\rho: \qquad \beta B_t E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] = \mu_t\beta E_t\left[\frac{P_t}{P_{t+1}}\right]$$

$$\partial P: \qquad \frac{\lambda_t}{P_t^2}\left(B_t - \rho_{t-1}B_{t-1}\right) = \mu_t\rho_t\beta E_t\left[\frac{1}{P_{t+1}}\right] - \mu_{t-1}\rho_{t-1}\frac{P_{t-1}}{P_t^2}$$

The equations with "∂" in front of them at the left are Euler equations of the government. The conditions (6)-(8) are treated as constraints by the government, except that the constraint $\lim_{t\to\infty} E\left[\beta^t B_t/P_t\right] \geq 0$ implied by (8) is the government's TVC. It prevents excessively rapid growth of *negative* government debt, i.e. the rapid growth of government ownership of private sector wealth. A perhaps more realistic alternative is the $B_t \geq 0$ condition. The other half of the limit constraint in (8), $\lim_{t\to\infty} E\left[\beta^t B_t/P_t\right] \leq 0$, might follow from a private sector TVC, and becomes a no-Ponzi condition for the government.

Algebra

From the B and ρ FOC's, together with the private FOC (7), we can derive a relation of μ_t to λ_t:

$$\mu_t = \frac{B_t}{P_t}\lambda_t \tag{9}$$

Using this and (7) again in the P FOC, we get

$$\lambda_t(B_t - \rho_{t-1}B_{t-1}) = \lambda_t B_t - \lambda_{t-1}B_{t-1}\rho_{t-1}\ ,$$

which reduces to $\lambda_t = \lambda_{t-1}$, and thus via the τ FOC to $\tau_t = \tau_{t-1}$.

Discussion

The conclusion we have derived here applies only *after* $t = 0$. This is because it uses, in time-t FOC's, expectational constraints dated $t - 1$. In particular, the FOC w.r.t. P_0 is not the P FOC given above with t set to zero, but instead

$$\partial P_0: \qquad \frac{\lambda_0}{P_0^2}\left(B_0 - \rho_{-1} B_{-1}\right) = \mu_0 \rho_0 \beta E_0 \left[\frac{1}{P_1}\right] . \qquad (10)$$

which reduces to $\lambda_0 \rho_{-1} B_{-1} / P_0^2 = 0$. We postpone momentarily discussing the implications of this initial-date condition. Also, we have assumed an interior solution. If there is a $B \geq 0$ no-Ponzi condition, and we were up against this constraint last period, the $\tau_t = \tau_{t-1}$ conclusion no longer applies. We postpone discussion of this temporarily also.

- The intertemporal budget constraint of the government, which applies because of the no-Ponzi condition and the TVC, requires that τ_0 be set to satisfy the same condition (5) as before with B/P replacing B.

- If τ is thus set to provide time-0 fiscal balance, how does the government maintain fiscal balance, with τ_t constant but future G's changing stochastically?

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- If τ is thus set to provide time-0 fiscal balance, how does the government maintain fiscal balance, with τ_t constant but future G's changing stochastically?

- It need "do" nothing. The price level will adjust automatically to maintain the fiscal intertemporal budget constraint.

- A surprise increase in expected future G's leads to a surprise increase in P just sufficient to reduce B/P by enough to maintain intertemporal budget balance.

The first period

[Note: The previous version of these notes included an incorrect discussion of optimal policy at time 0 in the absence of a $B \geq 0$ *contraint. There is an optimal policy in such a case, according to the logic of the model, assuming* $\ell(\tau)$ *is smooth and with a minimum at* $\tau = 0$*, but we do not discuss it in detail because the interpretation of the model is strained in this case.]*

The first period

At $t = 0$, optimal policy satisfies the P_0 FOC (10) above. For this to hold, with B_{t-1} and ρ_{t-1} non-zero, requires $\lambda_0 = 0$ or $P_0 = \infty$. $\lambda_t = 0$ is possible only at $\tau = 0$, where the marginal deadweight loss from taxation $\ell'(0)$ is by assumption zero. That in turn is possible only if debt is zero and current G is zero. To see this, note that if debt is issued at $t = 0$, then at $t = 1$ we will have $\tau_1 = \tau_0 = 0$, and this condition persists as long as $B_t > 0$. But with zero taxes and $G > 0$, there is no way that B can decrease, so this policy will produce exponential growth in real debt at the real interest rate and is therefore unsustainable.

First period with $\tau_0 > 0$

If $\tau_0 > 0$, then the P_0 FOC requires $P_0 = \infty$, which is not technically possible. If P is ever infinite, then the future rates of inflation that enter into the model's equation are all undefined. However, it is true that welfare is higher the higher the initial P, so that optimal policy if $B_{-1} > 0$ is in fact a very large initial surprise inflation that all but wipes out the real value of B_{-1}. The government can achieve this by issuing large amounts of debt — making B_0 very large. With a given future τ stream, B_0/P_0 is unaffected by the size of B_0, so the large B_0 requires large P_0. Since $\rho_{t-1}B_{t-1}$ is given, it shrinks in real value as the amount of nominal debt issue rises.

First period with previously issued debt inflated away.

If the real value of $\rho_{-1}B_{-1}$ has been inflated away, the time-0 budget constraint is

$$\frac{B_0}{P_0} = G_0 - \tau_0 \,.$$

But the initial real value of the debt must also match the discounted present value of future primary surpluses, and optimal policy requires $\tau_0 = \tau_1$ if $\tau_0 > 0$. This gives us

$$G_0 - \tau_0 = G_0 - \bar{\tau} = \frac{\beta\bar{\tau}}{1-\beta} - E_0\left[\sum_{t=1}^{\infty}\beta^t G_t\right]$$

and therefore $\bar{\tau} = (1-\beta)E_t\left[\sum_{t=0}^{\infty}\beta^t G_t\right] \,.$

What if this implies negative B_0?

If the value of $\bar{\tau}$ on the line above is greater than G_0, as would occur if G_0 were unusuallly low, then $B_0/P?0 = G_0 - \bar{\tau} < 0$. With the $B_0 \geq 0$ constraint this is impossible, and in fact in this case optimal policy is to set $B_0 = 0$ and $\tau_0 = G_0$. When the $B_0 \geq 0$ restriction is replaced by the looser no-Ponzi restriction, it is indeed optimal to make B_0/P_0 negative. However, in this case interpretation of the model is difficult — it implies the government can control the price level by the amount of nominal lending it does to the pubic.

With stochastic G, inflating away the debt can happen again

With stochastic G_t, the fact that $\bar{\tau}$ has been chosen to match *expected* discounted present value of G does not mean that it will be enough to cover G, even with debt inflated away, in every future period. When such a period arises, it will be optimal to inflate away essentially all the real value of the debt again, then set τ_t to a new, higher level, that will again be maintained constant until the next dire fiscal emergency that cannot be covered with an unexpected inflation tax on debt-holders.

Making these assertions rigorous requires specifying the stochastic properties of G_t. More detailed discussion of these "debt repudiation dynamics" are in my paper "Fiscal consequences for Mexico of adopting the dollar", on my website and on the course reading list.

Remarks about realism

Barro's model and its extension to consider inflation are both drastic simplifications, meant to illustrate a mechanism that should be taken seriously. Taxes should generally be smoothed, to some extent. Unanticipated inflation (and, in a model with long-term debt, unanticipated interest rate changes) are available as tools to smooth taxes. In reality unanticipated inflation has costs that are not incorporated in these models, because of price stickiness, money illusion, contracts written without inflation contingencies, etc. These have to be balanced off against the benefits of using unanticipated inflation to smooth taxes.

It also might be objected that the model's assumption that the government can announce future policies and be believed — i.e. can "commit"— is unrealistic. Note, however, that a government that really cannot commit can never issue any debt, because every period a government starting anew will want to repudiate debt issued by previous governments. Any interesting model of intertemporal fiscal policy must therefore assume some type of commitment ability on the part of the government. In fact, standard models of monetary policy without commitment can be regarded as internally contradictory, as they nearly always assume in the background a treasury that can issue debt.