LAND PRICE LINEARIZATION EXERCISE

Consider an economy in which the representative agent's utility is

$$E\left[\sum_0^\infty \beta^t \log C_t\right] \tag{1}$$

and the agent faces the constraint

$$C_t + Q_t L_t + P_t K_t = (Q_t + r_t)L_{t-1} + .98 \cdot (f_t + P_t)K_{t-1}\,, \tag{2}$$

where Q_t is the price of land at time t in terms of consumption goods, P_t is the price of capital at time t, r_t is the rental price of land, and f_t is the rental price of capital. The .98 factor reflects depreciation of capital. Assume the discount factor $\beta = .95$.

Production is carried out by representative firms that produce capital and consumption goods using rented land and capital and maximize profits. That is, they solve

$$\max_{I_t, C_t, \ell_t, k_t} P_t I_t + C_t - r_t \ell_t - f_t k_t \tag{3}$$

$$\text{subject to } \sqrt{C_t^2 + I_t^2} \le A_t \ell_t^{.5} k_t^{.5}\,, \tag{4}$$

where A_t is the level of technology, ℓ_t is land rented at t, I_t is new capital produced at t, and k_t is capital rented at t.

We assume that $\Delta \log A_t = .0002 + .99\Delta \log A_{t-1} + \varepsilon_t$, with ε_t i.i.d. and mean zero, so the growth rate of technology is a stationary, but very persistent, process with mean .02 per period. Market clearing conditions are

$$I_t = K_t - .98K_{t-1} \tag{5}$$

$$\ell_t = L_{t-1} = 1 \tag{6}$$

$$k_t = .98K_{t-1}\,. \tag{7}$$

Implicitly, by using the same symbol for per capita consumption in the firm and personal problem statements, we are imposing market clearing in consumption goods.

Show that the model has a deterministic constant growth rate solution with C/K constant.

The FOC's for the the representative agent are

$$\begin{aligned}
\partial C: &\qquad \lambda_t = \frac{1}{C_t} \\
\partial K: &\qquad P_t\lambda_t = .98\beta E_t[(P_{t+1} + f_{t+1})\lambda_{t+1}] \\
\partial L: &\qquad Q_t\lambda_t = \beta E_t[(Q_{t+1} + r_{t+1})\lambda_{t+1}]
\end{aligned}$$

For the firm they are

$$\begin{aligned}
\partial I: &\qquad P_t\zeta_t = \frac{\mu_t I_t}{Y_t} \\
\partial C: &\qquad \zeta_t = \frac{\mu_t C_t}{Y_t} \\
\partial k: &\qquad f_t\zeta_t = \frac{.5\mu_t Y_t}{k_t} \\
\partial \ell: &\qquad r_t\zeta_t = .5\mu_t Y_t\,,
\end{aligned}$$

where $Y_t = \sqrt{C_t^2 + Y_t^2} = A_t k_t^{.5}\ell_t^{.5}$. To keep the algebra simpler, we introduce $g = e^{.04}$. The FOC's, the market-clearing conditions, and the social resource constraint

$$\sqrt{C_t^2 + I_t^2} = A_t K_{t-1}^{.5}$$

can be rewritten in terms of the new variables (a couple of which are the same as the old ones)

$$\begin{aligned}
c &= \frac{C}{K} \\
i &= \frac{I}{K} \\
k &= \frac{K}{\bar{K}g^t} \\
p &= P \\
q &= \frac{Q}{C} \\
f &= f \\
r &= \frac{r}{c} \\
a &= \log(A/\bar{A}) - .02t
\end{aligned}$$

The k defined here is not the same as the k that appeared in the producer's problem in the original system. From here on, the new definition of k applies.

The resulting equation system is

Krent : $f = 0.5\dfrac{(c^2+i^2)g}{0.98ck_{-1}/k}$

SRC : $\sqrt{c^2+i^2} = \exp(a)\bar{A}\dfrac{(0.98*k_{-1})^0.5}{k\bar{K}^{.5}g^0.5}$

Idef : $i = 1 - \dfrac{0.98k_{-1}}{g*k}$

Kreturn* : $\dfrac{i}{c} = E_t\left[0.950.98\dfrac{ck}{c_{+1}gk_{+1}}\left(\dfrac{i_{+1}}{c_{+1}}+f\right)\right]$

LandReturn : $q = 0.95E_t[q_{+1}+r_{+1})$

LandMP : $r = 0.5(1+p^2)$

PKdef : $p = \dfrac{i}{c}$

Adynamics : $\Delta a = .0002 + \Delta a_{-1} + \varepsilon$

These are eight equations in the eight unknowns given by the list of variables above. The system is written entirely in terms of the 8 variables listed above, except for $\bar{K}$ and $\bar{A}$. These two varibles enter only the social resource constraint (SRC). We can normalize $\bar{A}$ to 1. In steady state, k as we have defined it here is 1, automatically, so we can solve the steady state system (in which all leads and lags are set equal) for $\bar{K}$ and leave k out of it. The last equation, for a, becomes an identity in the steady-state system, which is why we can set $\bar{A}$ arbitarily.

Find that steady state.

Solving the system for steady state, we find (not quite what was circulated as the steady state in the hint, alas)

c	i	$\bar{K}$	f	q	r	p
0.0978	0.0584	72.5548	0.0705	12.8909	0.6785	0.5974

Linearize around the steady state.

Here are the contemporaneous and lagged coefficient matrices, ready for input into `gensys`;

g0

	c	i	k	f	q	r	p	A	z
Krent	−0.3415	−0.6345	−0.0010	1.0000	0.0	0.0	0.0	0.0	0.0
SRC	0.8585	0.5129	0.0016	0.0	0.0	0.0	0.0	−0.1139	0.0
Idef	0.0	1.0000	−0.0130	0.0	0.0	0.0	0.0	0.0	0.0
Kreturn	11.5737	−9.1467	0.0082	−0.8945	0.0	0.0	0.0	0.0	0.0
LandReturn	0.0	0.0	0.0	0.0	−0.9500	−0.9500	0.0	0.0	0.0
LandMP	0.0	0.0	0.0	0.0	0.0	1.0000	−0.5974	0.0	0.0
PKdef	6.1091	−10.2255	0.0	0.0	0.0	0.0	1.0000	0.0	0.0
Adynamics	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0
zdef	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

g1

	c	i	k	f	q	r	p	A	z
Krent	0.0	0.0	−0.0010	0.0	0.0	0.0	0.0	0.0	0.0
SRC	0.0	0.0	0.0008	0.0	0.0	0.0	0.0	0.0	0.0
Idef	0.0	0.0	−0.0130	0.0	0.0	0.0	0.0	0.0	0.0
Kreturn	12.2183	−10.2255	0.0082	0.0	0.0	0.0	0.0	0.0	0.0
LandReturn	0.0	0.0	0.0	0.0	−1.0000	0.0	0.0	0.0	0.0
LandMP	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
PKdef	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Adynamics	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9900	−0.9900
zdef	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0

The `Pi` matrix is two columns with ones in the positions for Kreturn and LandReturn an zeros elsewhere, and the `Pi` matrix is a single column with a one in the position for Adynamics only. The system introduces an artificial variable $z_t = a_{t-1}$ to keep it formally a first-order (in the sense of one lag) system. There are around 20 non-trivial derivative coefficients to calculate in these matrices. In most cases the formulas are not terribly complicated, but a few are complicated, and in making 20 calculations the chances of error are fairly high. The matrices displayed above were generated by "automatic differentiation". Automatic differentiation uses the code for an expression to generate code for the derivative of the expression. As was mentioned in the previous "hint" message, R code that does this for potentially nonlinear rational expectations systems is available on my web site.

Solve the linearized model, and calculate the impulse responses of C_t/K_t, I_t/K_t, Q_t and P_t to shocks to technology growth rates (ε_t). Does the model imply that there are wide swings in land and capital prices in response to technology growth shocks? Are land prices more volatile than capital prices?

The impulse responses can be found in R with this code (assuming the output of `gensys` is in the object `gout`:

```
ir <- matrix(0, 9, 40)
ir[ , 1] <- gout$impulse
for (it in 2:40) ir[ , it] <- gout$G1 %*% ir[ , it-1]
```

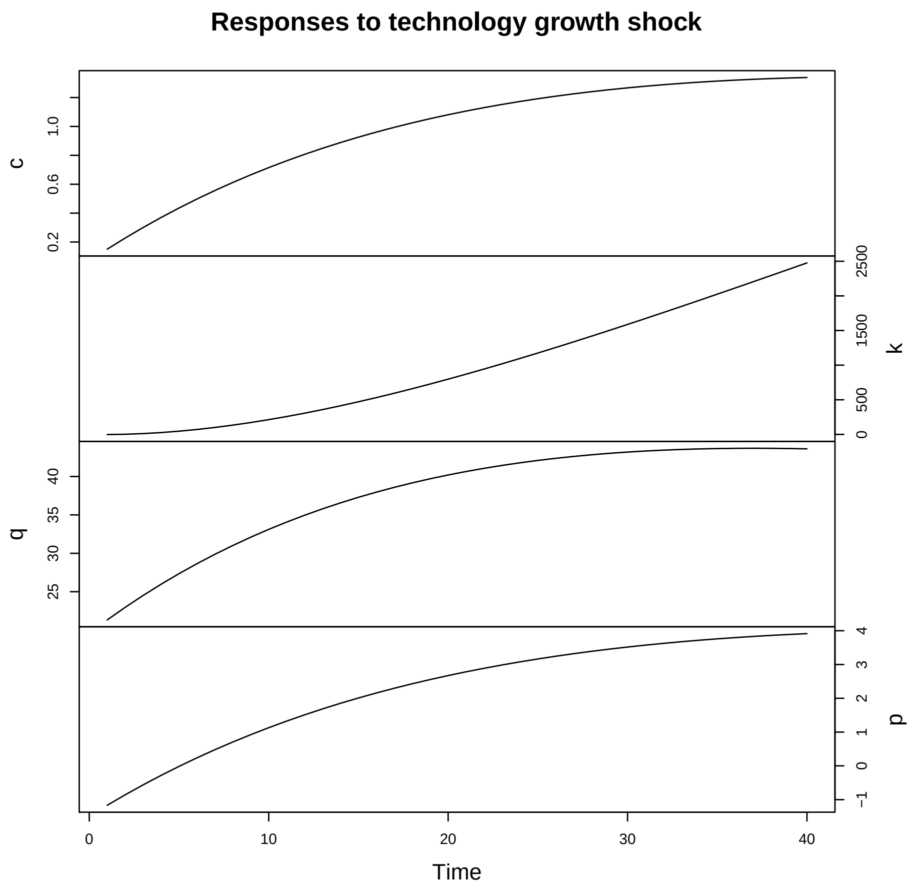
or in Matlab or Octave with this code (assuming the returned values from `gensys` all have their default names):

```
ir = zeros(9,40);
ir(:,1) = impulse;
for it = 2:40
   ir(:, it) = G1 * ir(:, it-1);
end
```

The plot of impulse responses below was produced with the R command

```
plot(ts(t(ir[c(c","k","p","q"), ])), yax.flip=TRUE, main="Responses to technology growth shock")
```

This selects the `c,i,p,q` rows of the impulse responses, transposes the resulting matrix, and plots it with the time-series-plotting function (selected automatically because the matrix has been converted to multiple time series by the `ts()` function).



From these plots we see that q, the ratio of Q to C, responds strongly positively, jumping up initially and then growing steadily when the technology growth rate jumps up. Since c, the ratio of C to K, also responds positively, Q itself responds even more positively. The price of capital, p, on the other hand, initially declines slightly, then rises.

To compare magnitudes, we need to take account of steady state values and convert to percentage changes. q's steady state is about 20 times as big as p's, k's is about 100 times as big, and c's is about 6 times smaller. The response of Q in percentage terms is the sum of the percentage responses of c, k, and q, which comes out at $t = 40$ to about 32 per cent for the same shock that makes p move by 4 per cent over that time span.

The intuition for this big difference is that capital is a reproducible asset, so when it becomes more useful, more of it is produced, limiting the increase in its market value, while land cannot be produced, so its increased productivity is reflected directly in its price. Indeed, because its substitute input, K, is increasing, the price of land increases both because of the direct effects of the technology shift and also because of the indirect effect from the increased ratio of capital to land.