EXERCISE ON NATURAL RESOURCES, LM'S, TVC'S

Consider the model with a representative agent who uses capital and an exhaustible natural resource in a constant-returns-to-scale technology to produce consumption and investment goods. Formally, the agent solves

$$\max_{C,K,R} E\left[\sum_{t=0}^{\infty}\beta^t U(C_t)\right] \quad \text{subject to} \tag{1}$$

$$C_t + K_t \leq A_t K_{t-1}^{\alpha} R_t^{1-\alpha} + K_{t-1} \tag{2}$$

$$S_t = S_{t-1} - R_t \tag{3}$$

$$S_t \geq 0 \tag{4}$$

$$A_t = A_{t-1}\varepsilon_t\,. \tag{5}$$

Here R_t is resources used up in period t, S_t is the stock of resources remaining after production at t, and C and K are as usual consumption and capital. A_t is the level of technology at t, and ε_t is an i.i.d., positive random variable with $E[\varepsilon_t] = 1$. The appearance of K on the right-hand side of (2) with a coefficient of 1, rather than $1-\delta$, means that there is no depreciation of capital.

(a) Note that if the stock of resources is ever exhausted (there is a t at which $S_t = 0$), no production is possible thereafter and consumption must shrink to zero after that. (More precisely, the sum of all consumption after that is finite.) Nonetheless if there is no shock ($\varepsilon_t \equiv 1$), it is feasible, given any initial capital stock K_0 and any initial stock of resources S_0, to choose paths of the choice variables so that $C_t > \bar{C}$ for all t, for some $\bar{C} > 0$. Prove this.

The statement of this part of the problem left out a key assumption: $\alpha > \frac{1}{2}$. Without this side condition it is still possible to make $K_t \to \infty$, but it is not possible to do so with C_t bounded away from zero. Solve the constraint for R_t, assuming C and A are constant:

$$R_t = \left(\frac{C + K_t - K_{t-1}}{AK_{t-1}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\,. \tag{$*$}$$

First note that if K shrinks monotonically from some date onward, then R_t is bounded below from that date onward and thus does not have a finite sum. Next suppose that $I_t = K_t - K_{t-1}$ is always positive, but converges to zero as $t \to \infty$. Then the constant C term in the numerator on the right of $(*)$ dominates in the limit and the whole expression behaves as

$$\left(\frac{C}{AK_t^{\alpha}}\right)^{1/(1-\alpha}\,. \tag{$\dagger$}$$

Now suppose K_t is, after some date t, equal to t^γ for some $\gamma > 0$. Then the expression above is $O(t^{-\gamma\alpha/(1-\alpha)})$. To be consistent with our assumption that $I_t \to 0$, we must have $\gamma < 1$. For $\sum R_t < \infty$ we require that the exponent on t in the denominator exceed 1. This will be true for some $\gamma < 1$ if $\alpha/(1-\alpha) > 1$, i.e. if $\alpha > 1/2$. So we have shown that there is a time path for K such that $\sum R_t < \infty$ when $\alpha > \frac{1}{2}$.

It is not hard to see that choosing $K_t = t^\gamma$ for $\gamma > 1$ will not relax the constraint on α. In this case we will have $I_t = O(t^{\gamma-1})$, and this makes $R_t = O(t^{\gamma-1/(1-\alpha)})$. This may make R summable if α is large enough, but with $\gamma > 1$ it will never make R summable for $\alpha \leq .5$.

This does not constitute a complete proof that $\alpha > \frac{1}{2}$ is necessary for maintaining C bounded away from zero, because we have considered only possible K paths of the form t^γ. However for the case $\alpha > \frac{1}{2}$ we have given an example of a path for k that makes $\sum R_t$ finite. If this particular path makes the sum exceed the initial endowment S_{-1}, we can simply multiply the C, I and K time paths by a constant less than one. As can be seen from $(*)$, this scales down the whole R path, and by scaling it down enough, we can make the sum of the R's less than S_0. Of course the scaling will not actually affect the initial K_{-1}. If we're not allowed to throw away capital, we can just in the first period $t = 0$ make C_0 large enough so that from then onward we are on the scaled down path.

(b) Derive the Euler equations and the TVC for this problem (*not* assuming $\varepsilon_t \equiv 1$).

We assume that the $S_t \geq 0$ constraint is never binding (since if it ever were, the objective function would be $-\infty$). We also treat the $S_t = S_{t-1} - R_t$ constraint as if it were a $\leq$ inequality, rather than an equality. This does not change the problem, since obviously we would never want to throw away resources and therefore would always keep the constraint binding. Euler equations:

$$\begin{aligned}
\partial C: &\qquad U'(C_t) = \lambda_t \\
\partial K: &\qquad \lambda_t = \beta E_t\left[\lambda_{t+1}\left(\alpha A_{t+1}\left(\frac{K_t}{R_{t+1}}\right)^{\alpha-1} + 1\right)\right] \\
\partial R: &\qquad \lambda_t(1-\alpha)A_t\left(\frac{K_{t-1}}{R_t}\right)^{\alpha} = \nu_t \\
\partial S: &\qquad \nu_t = \beta E_t \nu_{t+1}
\end{aligned}$$

The conditions for applying the standard TVC are met, including the non-negativity constraints that let us separate the TVC into a piece applying to S and a piece applying to K. They are

$$\begin{aligned}
\partial K: &\qquad E[\beta^t \lambda_t K_t] = E[\beta^t K_t U'(C_t)] \to 0 \\
\partial S: &\qquad E[\beta^t \nu_t R_t] = E[\beta^t (1-\alpha)(C_t + I_t)U'(C_t)] \to 0\,.
\end{aligned}$$

(c) Assuming $\varepsilon_t \equiv 1$, $U(C_t) = C^{1-\gamma}/(1-\gamma)$ (CRRA utility), $0 < \beta < 1$, $\gamma > 0$, and $0 < \alpha < 1$, show that no time path of the choice variables that maintains $C_t > \bar{C} > 0$ can be an optimum.

With this utility function, the K Euler equation can be rewritten as

$$\beta^{-1} = E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\gamma}\left(\alpha A_{t+1}\left(\frac{K_t}{R_{t+1}}\right)^{\alpha-1} + 1\right)\right].$$

To maintain a positive lower bound on C, we must prevent $K \to 0$, Yet to satisfy the resource constraint we must make $R_t \to 0$. Therefore the K_{t-1}/R_t term in the above equation goes to infinity, and the term itself is raised to the power $\alpha - 1 < 0$, so for large t, this expression says that C_t is expected to shrink at the rate $\beta^{1/\gamma}$. Obviously if $C_t > \bar{C} > 0$ for all t, this condition must eventually be violated.

(d) Can you propose a form for $U(C_t)$ that would make it possible for paths with $C_t > \bar{C} > 0$ to be optima, with the other assumptions as in part (c)?

The Euler equation requires that the marginal utility of consumption increase at an exponential rate eventually, as the rate of return to capital approaches zero. The CRRA utility forms can drive marginal utility to infinity exponentially only by driving C itself to zero exponentially. In applied work a common way to put a lower bound on C is to assume that there is a subsistence level of consumption C^* and that utility is $U(C_t - C^*)$. If U has the CRRA form, and $C^* > 0$, marginal utility will go to infinity as C_t approaches C^* from above. If we pick $\bar{C} > C^*$, the type of paths we derived in the first part of this problem will deliver a level of discounted utility greater than $-\infty$, whereas any path that fails to satisfy $C_t > C^*$ for all t will deliver minus infinity utility. The optimal path, whatever it is, must then deliver utility greater than minus infinity (since we know such paths exist) and therefore must keep $C_t > C^*$.

(e) Would the conclusions change if capital depreciated? if production were CES, i.e. $A_t(\alpha K_{t-1}^{\sigma} + (1-\alpha)R_t^{\sigma})^{1/\sigma}$ with $\sigma \neq 0$, $\sigma \leq 1$, instead of Cobb-Douglas?

With $\delta > 0$, at any given level of R, there is a maximum level of K, above which total production cannot cover depreciation, so that next period's K is less than this period's K even with $C = 0$. Thus it is impossible to keep K growing forever, and thus impossible to offset the necessarily declining R with increasing K. In a CES production function with $0 < \sigma < 1$, either of the two inputs (K and R) can be set to zero and output remains positive if the other input is non-zero. Thus in this case the resource can be exhausted (and optimally will be exhausted) in finite time, while C_t remains bounded away from zero forever. With $\sigma < 0$, at any given level of R there is an upper bound on output, and this upper bound shrinks linearly with R for large K. Thus it is not possible to maintain C by driving K up while R shrinks. These results reflect the fact that the isoquants of a CES production function cut the axes for $\sigma \in (0, 1]$, while they remain bounded away from the axes when $\sigma < 0$. For the Cobb-Douglas case $\sigma = 0$, they approach the axes asymptotically.

(f) Is it possible for ε_t to be stochastic and still preserve the conclusion that paths with $C_t > \bar{C} > 0$ are feasible?

Not with the assumptions we have made on ε_t. If ε_t is not identically one, then it has some probability of being less than some number $\phi < 1$. Though the probability may be small, there is then non-zero probability of arbitrarily many draws in a row of ε_t's less than ϕ. This means there is non-zero probability of A_t eventually being arbitrarily close to zero, and staying there arbitrarily long. Whatever our initially chosen $\bar{C}$, if A gets small enough and stays small long enough, we must eventually start shrinking K, and indeed K can therefore become arbitrarily small with non-zero probability. As $K \to 0$, the level of consumption that we can sustain forever goes to zero. So whatever $\bar{C}$ we tried to commit to, there is a non-zero probability that we will have such a string of bad luck that we will be forced to cut consumption below $\bar{C}$.