EXERCISE ON COINTEGRATION, GCP

(1) **Losing information from initial conditions** Consider the estimation of the mean μ from a sequence of N observations on X_t, which are drawn from a stationary Gaussian univariate process with autocovariance function

$$\frac{\sigma^2 \rho^{|s|}}{1-\rho^2} .$$

(X is an AR(1), in other words.) A crude way to estimate μ is to just take the sample average,

$$\bar{X} = \frac{1}{N} \sum_{t=1}^{N} X_t .$$

This is unbiased, but not efficient. It does happen to be asymptotically efficient because of the special character of the exogenous variable in the regression: a vector of ones.

Another simple asymptotically efficient estimator (assuming ρ is known) is to take the sample average of $(1-\rho L)X_t$ and divide by $1-\rho$:

$$\mu_{QD} = \frac{1}{(1-\rho)(N-1)} \sum_{t=2}^{T} X_t - \rho X_{t-1} .$$

(QD stands for "quasi-differenced".) While this is asymptotically efficient, it ignores information available in the first observation.

The fully efficient estimator is GLS, using the covariance matrix of the observations

$$\Omega = \frac{1}{1-\rho^2} \left[\rho^{|i-j|} \right] .$$

to construct the maximum likelihood estimator. We'll call this estimator μ_{GLS}, since it is a special case of generalized least squares estimation.

Calculate and plot, for samples of size 10, 100, and 1000, the ratios of the standard deviations of μ_{QD} and μ_{GLS} to the standard deviation of $\bar{X}$, as a function of ρ. Note that for $|\rho| \geq 1$, X_t cannot be a stationary process and thus has no fixed mean, while the interesting behavior of these ratios is likely to be for values of $|\rho|$ near 1.

[Why is $\bar{X}$ asymptotically efficient here? OLS is efficient whenever the $T \times k$ X matrix lies in the space spanned by k eigenvectors of Ω. The constant vector is not an eigenvector of Ω in this problem, but it is close to being one, and gets closer to being one as sample size increases.]

[It is possible to derive formulas for the variances of all three estimators as functions of sample size and ρ, but it may be easier to use the usual matrix expressions from regression model theory. If you do it that way, the `toeplitz` function in R or similar functions in other languages that construct a Toeplitz matrix may be useful, as Ω is Toeplitz.]

(2) **Granger Causal Ordering**Suppose a 3-variable VAR $(I - B(L))y_t = \varepsilon_t$ has a pattern of zeros in the $B(L)$ matrix polynomial. For each of the patterns below, determine whether the system has any Granger causal ordering and state which variables are causally prior to (GCP to) which.

$$a) \quad \begin{bmatrix} x & 0 & 0 \\ x & x & x \\ 0 & x & x \end{bmatrix} \qquad b) \quad \begin{bmatrix} x & 0 & x \\ x & x & x \\ x & 0 & x \end{bmatrix}$$

$$c) \quad \begin{bmatrix} x & x & 0 \\ 0 & x & 0 \\ 0 & x & x \end{bmatrix} \qquad d) \quad \begin{bmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{bmatrix}$$

(3) **Cointegration** Here is a bivariate second-order VAR system, $y_t = B(L)y_t + \varepsilon_t$, with

$$B_1 = \begin{bmatrix} 1.1 & 0 \\ 0.3 & 1.4 \end{bmatrix} \qquad B_2 = \begin{bmatrix} -.4 & -.2 \\ 0 & -0.2 \end{bmatrix} .$$

(a) Show the system is non-stationary.
(b) Show that it is cointegrated.
(c) Rewrite it in VECM form. In doing this you will also display the coefficients of the cointegrating vector — the coefficients in a linear combination of the elements of y that is stationary.