TAKE-HOME EXAM

This is an exam, not an exercise, so you are not to collaborate on it and you are not to discuss it with anyone else until after the time the exam is due (9AM Tuesday, May 22), even if you finish it earlier. It is open book, so you can consult your notes or any references you like. The exam should not take more then 3-4 hours, even allowing for time to think about it. The exam can be handed in electronically. If it is handed in on paper it should go directly to me in my office, 104 Fisher or slipped under the office door, assuming it is handed in on time. If you miss the deadline (which of course incurs a grade penalty), be sure you let me know by email to make arrangements for late submission.

Smith, Suchanek and Williams (SSW) in an article on the course reading list describe an experimental market with the following structure. Players in the market each have some initial amount of "cash" and some initial amount of an "asset". Different players may have different amounts. Over the course of T periods, agents can trade in a computer auction market. At the end of each period t, the asset pays a dividend of δ_t per unit to those owning the asset. When the experiment is over, the players turn in their cash and their assets for actual money.

Though the SSW setup had some variants, we will consider this one version: δ_t is one dollar with probability .5, zero with probability .5, and these probabilities are independent across periods and known to all the players. The asset pays *only* the dividends, so after the dividend payout at T (if any) it is worthless.

There is a "fundamental" price for the asset: $\bar{P}_t = .5N_t$, where $N_t = T - t + 1$ is the number of dividend payout periods remaining (.i.e. T during the first period of trading, 1 during the last period of trading.

(1) Explain why, if the market is competitive and the players all rational, the fundamental price is the one we would expect to prevail. Make your assumptions explicit. Proving this carefully may be harder than you initially think. One good approach is to make the argument first for the last period T, then make it again at $T - 1$ assuming that it holds at T, etc.

Simply saying that price should be equal to expected future dividend returns is not a complete answer. If there are expected capital gains or losses, they also can affect the price. In the last period, though, it is known that the "next period" price will be zero. So only the dividend matters, and if the price

Date: May 26, 2007.

is below .5, rational risk-neutral investors will want to trade zero-return cash for the asset, while if the price is below .5 they will want to do the reverse. Since all the cash and assets in the economy have to be held by someone, the price must be set to make everyone indifferent between them, i.e. to make expected return zero on the asset, as it is on cash. Then in the period before the last, the absolute expected return will be the expected dividend of .5, plus the next period value of one unit of the asset, which is also .5. Thus to preserve the zero expected return the price at $T-1$ must be \$1. Continuing recursively backwards, this leads to the $N_t/2$ formula.

SSW note that in most of their experiments, when the players were not highly experienced, asset prices showed a bubble and crash pattern. The price P_t started out below the fundamental price $\bar{P}_t$, rose to a peak above $\bar{P}_t$, then dropped quickly back to $\bar{P}_t$. Suppose (as in some of the SSW experiments) $T = 30$. We want to consider the behavior of a rational player who knows that the other players are not rational and are therefore likely to produce a bubble and crash. To keep things simple we assume that $P_1 = 7$, and that at each period $t < T$ there is a probability π_t of a "crash". When there is a crash at t, $P_t = \bar{P}_t$ and $P_{t+s} = \bar{P}_{t+s}$ for all $s > 0$. If there is no crash at t, $P_t = P_{t-1} + 1$. There is no chance of a crash at $t+1$ so long as $P_t < \bar{P}_t$, but after $P_t \geq \bar{P}_t$ for the first time there is a probability $\pi_t = 2/N_t$ each period that next period will be a crash. (Recall from above that $N_t = 30 - t + 1$ is the number of remaining dividend payouts.) Note that when $t = 29$, $2/N = 1$, so the asset value returns to fundamental value in the last period if not before.

In other words, this is a situation where price will surely move upward until it equals or exceeds fundamental price, then it will proceed above the fundamental price with some probability, and keeps increasing until a crash occurs. There will be just one crash, so that the price will rise to a single peak, then return to the fundamental path. Summarizing before-crash price behavior,

$$P_{t+1} = \begin{cases} P_t + 1 & \text{with probability 1 if } P_t < \bar{P}_t \\ P_t + 1 & \text{with probability } 1 - 2/N_t \text{ if } P_t \geq \bar{P}_t \\ \bar{P}_{t+1} & \text{with probability } 2/N_t \text{ if } P_t \geq \bar{P}_t\,. \end{cases}$$

(2) In SSW, players can hold only non-negative amounts of the asset and of cash (no short selling or borrowing). Explain why a rational player who is not risk averse, knowing the price behaves as described above, will hold only the asset, using all his cash to buy the asset, so long as $P_t < \bar{P}_t$.

If the asset is sure to appreciate by one, and it also has an expected dividend yield of .5, it has positive expected yield and thus dominates cash in expected return. A risk-neutral rational agent will therefore hold his entire portfolio in the asset.

(3) Show that this non-risk-averse, rational player will not sell the asset immediately when the price rises above the fundamental value.

Once the price rises above the fundamental value, there is a possibility of a capital loss. Expected absolute return is therefore $\pi_t(\bar{P}_{t+1} - P_t) + (1 - \pi_t) * 1 + .5$. This is positive unless $\bar{P}_{t+1}$ is much less than P_t, which makes it likely that when price first goes above $\bar{P}_t$, expected return will still be positive. The price first goes above fundamental value in period 7, when the price is 13 and the fundamental price is 12. The small probability of this $1 loss is not enough to make the return negative, so the rational agent does not sell.

(4) Show that this non-risk-averse, rational player will certainly sell his entire holdings of the asset at a certain date before the crash, if the crash occurs late enough. Calculate the date at which he will sell.

These computations are done on the accompanying spreadsheet. The expected absolute return becomes slightly negative at period 14 if the crash has not occurred before then,so the rational agent will convert to cash at that point.

(5) Calculate the maximum and the minimum amount he will have at the end of play. Which crash date is worst for him? Which best?

He will hold the asset in periods 1-13, convert to cash at period 14, then stay entirely in cash until the crash. After the crash expected return will be the same on the asset as on cash, so he may hold either. By period $T = 30$, the crash will definitely have occurred. He might therefore put all his wealth into the asset. The asset will be priced at 50 cents, and he will have a 50% chance of getting dividends of 1$, i.e. of doubling his money. Of course he also has a 50% chance of losing everything which, being risk-neutral, he is willing to contemplate. But this means that no matter what happens before $T = 30$, it is possible for this agent to end up with no wealth at all, because he might risk it all on a last-period double-or-nothing bet.

On the other hand, the best possible outcome for him is that he first"rides the bubble", making high returns, all the way to period 14 and that also all this time the asset happens to pay out 1 per period, not 0. He will definitely not buy any of the asset in period 14, but if the crash occurs immediately, he might again buy it (he will be indifferent between it and cash) in period 15. If from then on the asset pays a dividend of 1 each period, he gets the maximum possible total return. It turns out that this will leave him with $85 for every dollar of initial wealth — i.e. an 8,500% return. However this requires dividends of \$1 in each of 29 periods, in which they could with equal probability be zero. The odds against this are approximately 537 million to 1. An interesting question not exactly asked by the exam is how *expected* total return depends on the crash date. Because he earns positive returns until the date at which he switches to cash, expected total return is highest if the crash comes after he has switched to cash, i.e. after $t = 14$. After that date expected return on the asset and on cash is zero, so from the point of view of expected return it does not matter to him at what date the crash occurs, so long as it is after $t = 14$. In expected value, he ends up with \$3.29 for every \$1 of initial wealth, i.e. a 329% return. If the crash occurs between $t = 7$ and $t = 14$, its effects are worse the later the crash, with the worst expected total return for a crash at $t = 14$, producing total expected return 1.56 per invested dollar. Note that this is more than the ratio of $\bar{P}_{14}$ to P_1, because the player has been accumulating and reinvesting dividends.

(6) Suppose that instead of there being a single rational agent like this, everyone in the market is non-risk-averse and thinks that there will be a bubble and crash as described above (though possibly with $P_1 \neq 7$). Could their beliefs be self-confirming, so that the bubble and crash then actually occur?

Most people missed this one. All of the assets and cash must be held by someone. These rational-but-mistaken agents will believe the expected return on assets is positive, and thus want to hold only the asset, unless the price is driven very high. In the first period, the price at which the asset expected return is zero, making all the agents indifferent between the asset and money, is $P_1 = 28.5$. However, since next period, if there is no crash, the price will be even higher (and thus the possible capital loss with a crash greater) and the probability of a crash even greater, they will all plan to sell at $t = 2$, triggering an actual crash at that date. So the conclusion is that rational, risk-neutral agents with these beliefs would pay a higher-than-fundamental initial price, but that they would force an immediate crash to the fundamental price in the next period. Of course this would not prove to the agents that they were wrong — it would look to them as if they just happened to have bad luck — the crash, despite having probability only 2/29 in period two, unfortunately occurred then. This equilibrium is in a sense a bubble and crash equilibrium, but the “bubble” is trivial; there is no initial period of rising prices and capital gains.